Maturity, Indebtedness and Default Risk [1]

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Abstract

In this paper, we present a tractable model of long-term sovereign debt and explore its implications for levels of sovereign debt and default spreads. Using Argentina as as test case, we show that our model of long-term sovereign debt is capable of explaining both the average spread (or, equivalently, default probability), the average debt-to-output ratio and the volatility of consumption relative to output during the ten years preceding Argentina's most recent default episode (in 2002). We show that our model of long-term sovereign debt improves upon models with short-term debt in that short-term sovereign debt models seem incapable of explaining all three facts for Argentina. We also use our model to answer a counterfactual: What would debt levels, spreads, consumption volatility and be if Argentina had issued only short-term debt?

Key Words: Unsecured Debt, Sovereign Debt, Long Duration Bonds, Debt Dilution, Random Maturity Bonds, Default Risk

JEL:

1 Introduction

We study an equilibrium model of unsecured debt and default in which borrowers can issue long-term debt. The existing literature on this subject – both the consumer debt and sovereign debt parts – has mostly considered one-period debt. In reality, both consumers and countries can and do borrow long-term. For instance, in the US, the standard credit card contract requires borrowers to pay only a portion of their debt each period. Incorporating long-term debt brings these models closer to reality and, we argue in this paper, enhances their ability to account for observed high levels of household and sovereign debt and high default rates (spreads).

In existing quantitative models of one-period unsecured debt – both the consumer and the sovereign debt varieties – imply that default spreads rise very sharply with debt. Consequently, borrowers in the model simply do not wish to borrow much and these models tend to predict a small amount of debt and low spreads. Thus, these models have difficulty matching observed high levels of indebtedness and spreads. We show that lengthening the maturity of debt (i.e., going beyond one-quarter debt) can decrease the elasticity of spreads with respect to debt and thereby increase equilibrium borrowing as well as equilibrium spreads.

The main reason why this is possible is as follows. In our model, borrowers are required to repay a constant fraction of the existing debt each period and if they choose to not repay, they must stop repayment (i.e. default) on all existing debt (i.e., all debt is equally senior). Consider the situation of a borrower who is contemplating issuing an additional unit of debt. If income declines next period and the default risk on the additional new unit of debt rises, servicing the additional unit will not require the borrower to refinance the *entire stock* of debt tomorrow at the higher interest rate. Thus the decline in consumption from servicing the additional unit of debt when income is low tomorrow is not as sharp as in the case of one-period debt. This attenuates the incentive to default on the additional unit of debt relative to the one-period debt case and accounts for the decreased elasticity of spreads with respect to debt.

However the decreased elasticity of spreads comes at a cost to the lender. Even low levels of debt have a positive spread (which is costly). This happens because of the well-known "debt dilution" problem. The borrower cannot commit to not issue additional debt in the future and the lender must recognize that even if the current level of debt is very low, future levels of debt could be high. Thus (given the assumption that all debt is equally senior regardless of when it was issued) there is a positive probability of default even if total debt issued is small.

These two effects of longer-term debt tend to offset each other in different ways for different levels of debt. For low levels of debt, the cost of borrowing is actually higher for long maturity debt compared to one-period debt. But because the elasticity of spreads with respect to debt is low for long maturity debt, the interest rates for long maturity debt at high debt levels can be lower compared to one-period debt. If income is sufficiently volatile, the average level of equilibrium debt and equilibrium default risk (or spreads) can be considerably higher for long maturity debt compared to one-period debt.

Our paper relates to two branches of the unsecured debt literature. One branch is the literature on unsecured consumer debt. Chatterjee *et al* (2007) and Livshits, McGee and Tertilt (2007) – building on the earlier study by Athreya (2002) – are studies that attempt to account for the aggregate debt and default statistics for the US. These models employ short-term debt – more precisely, debt that matures in one model period. The framework presented in this paper is most closely aligned to the environment studied in Chatterjee *et al* and represents a significant extension of it. We provide a fairly complete characterization of the equilibrium bond pricing function as well as equilibrium behavior with regard to default and debt. These characterization results are used in the computation of the equilibrium. However, our current effort is less ambitious than Chatterjee *et al* in that we do not embed our model of long-term unsecured debt in a full-fledged general equilibrium model with incomplete markets in the style of Aiyagari (1994) or Hugget (1993).

The other branch is the literature on sovereign debt and default. The seminal contribution here is Eaton and Gersovitz (1981). More recently, Aguiar and Gopinath (2006) and Arellano (2007) have used the Eaton-Gersovitz framework to quantitatively account for some of the patterns regarding debt, interest rates, and default observed for emerging market economies. These authors have focused on the cyclical properties, in particular, on the fact that for emerging markets borrowing is positively related to output (the trade balance worsens in good times) and borrowing and interest rates are negatively related. Also, the volatility of consumption tends to be higher than the volatility of output, in marked contrast to (open) developed economies. Both studies assume that all debt is one-period (as in the original Eaton-Gersovitz model). In contrast, the focus of our paper is on first moments, in particular, on accounting for the average level of the default spread and the average level of the debt-to-income ratio. Existing studies have either not attempted or been unable to match these statistics. The contribution of our paper is to show that incorporating long-term bonds makes it possible to match both statistics easily. Furthermore, we show that a one-period debt model cannot simultaneously account for the debt-to-income ratio, the average spread, and the volatility of consumption relative to output. These points are demonstrated for the case of Argentina which has been the focus of previous studies.

There is a related literature on sovereign debt that attempts to go beyond one-period debt. Bi (2006) and Arellano and Ramnarayan (2008) focus on maturity choice and on the relationship between interest rate on long-term and short-term debt. The study that is most closely related to ours in Hatchondo and Martinez (2008). These authors introduce long-duration bonds into Arellano-style sovereign debt model. Their motivation is to improve upon the volatility of spreads predicted by Arellano, which is too low relative to the data. In contrast, as noted above, the focus of our paper is on first moments. There is also a difference in methodology. As we explain later in the paper, lengthening the maturity of bonds in a standard way leads to a computationally intractable model. Therefore, some "trick" is needed to analyze long-term bonds. Martinez and Hatchondo propose one way to do it and we propose another. Both strategies rely on making the long-term bond "memoryless" so that it is not necessary to keep track of the date-of-issue of the bond.

The paper is organized as follows. In section 2 we briefly discuss why incorporating long-term debt in the standard way into models of unsecured debt can lead to computationally intractable models and then describe our strategy for circumventing this problem. In section 3 we introduce the sovereign debt environment we analyze. In section 4 we present some characterization results for the equilibrium pricing function and sovereign decision rules. These results aid in the computation of the equilibrium of the model. Section 5 discusses some relevant computational issues. It turns out that grid-based algorithms for computing equilibrium models of default can easily encounter convergence problems. And these problems are compounded when there is long-term debt. These difficulties, and the way the are addressed in this paper, are explained in section 5. Section 6 presents the results of incorporating long-term debt for Argentina and explain how long-maturity bonds help explain the facts relative to one-quarter debt models. Section 7 concludes.

2 Modeling Long-Term Debt

A natural way to introduce long-term debt is to assume that debt issued in period t is due for repayment in period $t + T$. Since new debt can be issued each period this means that the issuers state vector contains the vector $(b_0, b_1, b_2, \ldots, b_{T-1})$ where b_τ is the quantity of bonds due for repayment τ periods in the future. Then, the probability of default on a bond due for repayment in period τ is the sum of the probability of default in the current period plus the probability of repayment in the current period but default in the next period plus the probability of repayment in the next two periods followed by default in the third period and so on all the way to period τ in the future. Even for modest values of T (such as 3 or 4), these calculations can become quite demanding because we will have at least T state variables each of which can potentially take many, many values.

Our approach is to simplify the maturity structure of debt in a way that calculation of default probabilities from the individual's decision problem becomes easier. We analyze long-term debt contracts which mature probabilistically. Specifically, each unit of outstanding debt matures next period with probability λ. If the unit does not mature – which happens with probability $1 - \lambda$ – it gives out a coupon payment z. Observe that if $\lambda = 1$ then the bond is a one-period discount bond and if $z > 0$ and $\lambda = 0$ then the bond is a consol promising to pay z units each period. For intermediate values of λ we have a bond which matures, on average, in $1/\lambda$ periods.

Why is this probabilistic maturity structure easier to analyze? The benefit comes from the "memory-less" nature of the bond. Going forward, a unit bond of type (z, λ) issued $k \geq 1$ periods in the past has exactly the same payoff structure as another (z, λ) unit bond issued $k' > k$ periods in the past. This means we can aggregate all outstanding (z, λ) unit bonds regardless of the date of issue and thereby cut down on the number of state variables relevant to the individual's decision problem. This reduction in turn reduces the computational burden of computing default probabilities.

In what follows we will assume that unit bonds are infinitesimally small – meaning that if b unit bonds of type (z, λ) are outstanding at the start of next period, the issuer's coupon obligations next period will be $z \cdot (1-\lambda)b$ for sure and her payment-of-principle obligations will be λb for sure. And, if no new bonds are issued or no outstanding bonds redeemed next period, $(1-\lambda)b$ unit bonds will be outstanding for sure at the start of the following period.

Hatchondo and Martinez (2007) use a similar trick of rendering outstanding obligations "memory-less" in order to analyze sovereign debt that lasts more than one (model) period. In their set up all bonds last forever (consols) but each pays a geometrically declining sequence of coupon payments. Thus, a bond issued in the current period promises to pay the sequence $\{1, \delta, \delta^2, \delta^3, \ldots\}$. One period later, the promised sequence of payments is $\{\delta, \delta^2, \delta^3, \delta^4, \ldots\}$ which is no different than the promised sequence on δ units of the bond issued last period. It is as if $(1-\delta)$ fraction of outstanding bonds mature each period and there is a coupon payment of 1 on every outstanding bond, including the ones that mature.

3 Environment

3.1 Preferences and Endowments

Time is discrete and denoted $t \in \{0, 1, 2, ...\}$. The sovereign receives a strictly positive endowment y_t each period. The stochastic evolution of y_t is governed by a finite-state Markov chain with state space $Y \subset \mathbb{R}_{++}$ and transition law $\Pr\{y_{t+1} = y' | y_t = y\} = F(y, y'),\;\; y$ and $y' \in Y$.

The sovereign maximizes expected utility over consumption sequences, where the utility from any given sequence c_t is given by:

$$\sum_{t=0}^{\infty} \beta^t u(c_t - m_t),\; \beta < 0 \tag{1}$$

The momentary utility function $u(\cdot) : [0, \infty) \to \mathbb{R}$ is continuous, strictly increasing, and strictly concave. The term $m_t \in M = [0, \bar{m}]$ is a minimum consumption requirement drawn independently each period from a probability distribution with continuous cdf $G(m)$.

The presence of the minimum consumption shock deserves some comment. It is introduced for two reasons. One reason is to make it possible for the model to account for the high volatility of consumption expenditures relative to output in emerging market countries. For instance, in the case of Argentina, the relative volatility exceeds 1. As we show in this paper, long maturity debt permits the sovereign to smooth consumption more effectively which tends to lower volatility of consumption relative to output. In the absence of a preference shock of some type, the volatility of consumption relative to output will tend to be too low in our model. The fact that shock is taken to be a shock to minimum consumption is not particularly important and other types of preference shocks would work as well. A second reason is to make robust computation of the model possible. In this regard, it is important that the shock be drawn from a continuous distribution and that it be i.i.d. The role played by these two assumptions are discussed later. It is worth pointing out that the minimum consumption requirement setup is isomorphic to a setup where there is no minimum consumption requirement but there is temporary i.i.d. shocks to endowments. This is the environment analyzed in Chatterjee *et al* so the results reported later apply to that environment as well.

3.2 Option to Default and the Market Arrangement

The sovereign can borrow in the international credit market and has the option to default on a loan.[1] If the sovereign defaults she cannot borrow in the period of default and, in the future, she is excluded from the international credit market for a random length of time. Specifically, upon default the sovereign is permitted to borrow in the future with probability $0 < \xi < 1$ and once she is permitted to borrow she can borrow in all subsequent periods until she defaults again. The periods in which the sovereign is excluded from borrowing – including the period of default – she loses some amount $\phi(y) > 0$ of her output y. In addition, in the period of default, the sovereign's minimum consumption requirement rises to its maximum value $\bar{m}$. We will assume that $y - \phi(y) - \bar{m} > 0$ for all y which ensures that discretionary output (total output less minimum consumption) is always strictly positive under all circumstances.

[1] For simplicity we rule out lending on part of the sovereign. In our quantitative work we will consider the situations in which β is low enough so that the constraint on lending is never binding.

As noted in the previous section, a bond in this economy is denoted by a pair of numbers (z, λ). We will assume that lenders are risk-neutral and the market for sovereign debt is competitive. The unit price of a bond of size b is given by $q_{z,\lambda}(y, b)$. Note that the unit price does not depend on the transitory shock m because knowledge of current period m does not help predict either m or y in the future and, therefore, does not inform the likelihood of future default. We will assume that the sovereign can choose the size of her bond issues from a finite set $B = \{0, b_1, b_2, \ldots, b_N\}$, where $0 > b_1 > b_2 > \ldots > b_N$.

3.3 Decision Problem

Consider the decision problem of a sovereign with $b \in B$ of type (z, λ) bonds outstanding, endowment y and minimum consumption requirement m. Because the sovereign is indebted, she has the option to default. Denote the sovereign's lifetime utility conditional on repayment, i.e. maintaining access to international credit markets, by the function $V(y, m, b) : Y \times M \times B \to \mathbb{R}$ and her lifetime utility conditional on being excluded from international credit markets by the function $W(y, m) : Y \times M \to \mathbb{R}$.

Then:

$$W(y, m) = u(c - m) + \beta\{[1 - \xi]E_{(y'\,m')|y}W(y', m') + \xi E_{(y'\,m')|y}V(y', m', 0)\} \tag{2}$$

s.t.

$$0 \leq c \leq y - \phi(y)$$

The sovereign's life-time utility under exclusion reflects the possibility that she may be let back into the credit market with probability ξ. If she is not let back, her situation next period will be the same as it is in the current period under exclusion – she loses $\phi(y)$ of her output and can expect to be let back into the credit market next period with probability ξ. In the period of default, the sovereign's lifetime utility is $W(y, \bar{m})$. Since $u(.)$ is strictly increasing, it is optimal for a sovereign who defaults or is excluded from international markets due to a prior default to simply consume all her available output.

And,

$$V(y,m,b) = \max_{b'} u\left(c-m\right) + \beta E_{(y'\, m')|y} \max\left\{V(y',m',b'), W(y',\bar{m})\right\} \tag{3}$$

s.t.

$$0 \leq c \leq y + [\lambda + [1-\lambda]z]\, b - q_{z,\lambda}(y,b')\left[b' - [1-\lambda]b\right]$$

The above implicitly assumes that the budget set under repayment is non-empty, meaning, there is at least one choice of b' which leads to non-negative discretionary consumption. But it is possible that (y,b,m) is such that all choices of b' lead to negative discretionary consumption. In this case, repayment is simply not an option and the sovereign must default. But in order to ensure that $V(.)$ is defined over the entire domain, we will assume that if the budget set is empty then

$$V(y,m,b) = u(0) + \beta\{[1-\xi]E_{(y'\, m')|y}W(y',m') + \xi E_{(y'\, m')|y}V(y',m',0)\} \tag{4}$$

Observe that in this definition the future looks exactly the same as under default. Thus, with this definition it is strictly optimal for the sovereign to choose default over "repayment" when the budget set under repayment is empty. This is so because the sovereign can get strictly positive consumption by declaring default – recall that by assumption $y - \phi(y) - \bar{m} > 0$.

We will assume that if the sovereign is indifferent between repayment and default, she repays. Hence, the country will default on debt b if and only if $W(y,\bar{m}) > V(y,m,b)$.

This decision problem implies a default decision rule $d(y,m,b)$, where $d = 1$ implies default and $d = 0$ implies repayment, and, conditional on repayment, a debt choice rule $a(y,m,b)$.

4 Equilibrium

The world one-period risk-free rate r_f is taken as exogenous. Given a competitive market in sovereign debt, the unit price of a bond of size b – $q_{z,\lambda}(y,b')$ – must be consistent with zero profits adjusting for the probability of default. That is:

$$q(y, b') = E_{(y'\, m')|y}\left[[1 - d(y', m', b')]\frac{\lambda + [1-\lambda][z + q(y', a(y', m', b'))]}{1 + r_f}\right] \quad (5)$$

Observe that in the future states in which the sovereign defaults the creditors get nothing. In the absence of default, the creditors get λ, which is the fraction the unit bond that matures next period, and, on the remaining fraction, the creditors get the coupon payment z. In addition, the fraction that does not mature will have some value next period that depends on the sovereign's endowment and borrowings next period.

5 Characterization of Equilibrium

In this section we characterize the equilibrium. These characterizations provide some intuition on the nature of the model and aid in the computation of the equilibrium. They are all in the nature of monotonicity results.

Proposition 1: There exist unique continuous and bounded functions $W(y, m)$ and $V(y, m, b)$ that solve the functional equations (2)- (4). In the region of the domain where repayment is feasible, $V(y, m, b)$ is strictly increasing in b and strictly decreasing in m.

Proof: The existence of unique, continuous and bounded solutions to the functional equations follow from standard Contraction Mapping arguments that will not be repeated here.

With regard to the mononicity of V, observe that if $m^0 < m^1$, then every b' that is feasible under (y, m^1, b) is also feasible under (y, m^0, b) and yields strictly higher discretionary consumption. Since the value of m does not affect the probability distribution of (y', m'), it follows that $V(y, m^0, b) > V(y, m^1, b)$.

Next, observe that if $b^0 < b^1$ then for every $b' \in B$ and every $y \in Y$ we have $[\lambda + [1-\lambda]z]b^0 + q(y, b')[1-\lambda]b^0 < [\lambda + [1-\lambda]z]b^1 + q(y, b')[1-\lambda]b^1$. This follows because $[\lambda + [1-\lambda]z] > 0$ and $q(y, b') \geq 0$. Hence, any b' that is feasible under (y, m, b^0) is also feasible under (y, m, b^1) and affords strictly greater discretionary consumption. Therefore $V(y, m, b^0) < V(y, m, b^1)$.

$\square$

The next proposition establishes that default is at least as likely under a higher debt level as under a lower debt level.

Proposition 2: If $b_0 < b_1$ then $d(y, m, b^0) \geq d(y, m, b^1)$.

Proof: Suppose, to get a contradiction, that $d(y, m, b^0) < d(y, m, b^1)$. Then it must be the case that $d(y, m, b^0) = 0$ and $d(y, m, b^1) = 1$. The former implies that $V(y, m, b^0) \geq W(y, \bar{m})$ and the latter implies $W(y, \bar{m}) > V(y, m, b^1)$. Then, we must have $V(y, m, b^0) > V(y, m, b^1)$. But this contradicts Proposition 1. Hence, $d(y, m, b^0) \geq d(y, m, b^1)$.

$\square$

One would expect Proposition 2 to imply that the equilibrium pricing function, $q(y, b')$, is increasing in b' (or, equivalently, that the equilibrium default premium is increasing in the level of indebtedness). In the case of one-period bonds this is indeed true. If $\lambda = 1$ the equilibrium pricing equation (5) reduces to $q(y, b') = E_{(y'\, m')|y}[1 - d(y', m', b')]/[1 + r_f]$ which is increasing by Proposition 2. But when $\lambda < 1$ (bonds have maturity longer than 1 period), the pay-off under repayment depends on the value of the outstanding bonds next period which, in turn, depends on the sovereign's output next period and the sovereign's borrowing decision next period. The following Proposition establishes that if the decision rule $a(y, m, b)$ is increasing in b then $q(y, b')$ is increasing in b'.

Proposition 3: Suppose $\lambda \in [0, 1)$. If $a(y, m, b)$ is increasing in b then the equilibrium pricing equation (5) implies that $q(y, b')$ is increasing in b'.

Proof: Let $d^*(y, m, b)$ and $a^*(y, m, b)$ be the equilibrium decision rules corresponding to the equilibrium pricing function $q^*(y, b')$. For any function $q : Y \times B \to \mathbb{R}_+$, define the operator $T(q)(y, b')$ as

$$E_{(y'\, m')|y}\left[[1 - d^*(y', m', b')]\frac{\lambda + [1 - \lambda][z + q(y', a^*(y', m', b'))]}{1 + r_f}\right].$$

Then the equilibrium pricing function $q^*(y, b')$ solves the equation $q^*(y, b') = T(q^*)(y, b')$.

Next, we will show that the operator T is a Contraction Mapping. Observe that (i) $q^1(y, b') \geq q^0(y, b')$ implies $T(q^1) \geq T(q^0)$ and (ii) for any positive constant θ and any $q(y, b')$, $T(q + \theta) \leq T(q) + \theta[1 - \lambda]/[1 + r_f]$ (this follows because $1 - d^*(y', m', x) \leq 1$). Since $[1 - \lambda]/[1 + r_f] < 1$, the operator T satisfies Blackwell's sufficiency conditions for a Contraction Mapping.

Next, we will show if $q(y, b')$ is a increasing function of b' then $T(q)(y, b')$ is also a increasing function of b'. Fix y, y' and m'. Let ${b'}^0 < {b'}^1$. Proposition 2 implies that $(1 - d^*(y', m', {b'}^0) \leq (1 - d^*(y', m', {b'}^1)$. Since $a^*(y, m, b')$ is increasing in b' by hypothesis and $q(y, b')$ is increasing in b' by assumption, it follows that $q(y', a^*(y', m', {b'}^0)) \leq q(y', a^*(y', m', {b'}^1))$. Since y' and m' was arbitrary, it follows that $T(q)(y, {b'}^0) \leq T(q)(y, {b'}^1)$.

Finally, to establish the result, let $\bar{q}$ be any number such that $q^*(y, b') \leq \bar{q}$ and let Q be the set of all non-negative functions $q(y, b')$ that are increasing in b' and bounded above by $\bar{q}$. Define the norm of any function $q \in Q$ as $||q|| = \sup q(y, b')$. Then $(Q, || \cdot ||)$ is a complete metric space. By the previous step, $T(q) \in Q$ for any $q \in Q$. Since T is a Contraction Mapping it follows from the Banach Contraction Mapping Principle that there exits a unique $\hat{q}(y, b') \in Q$ such that $T(\hat{q}) = \hat{q}$. But then $q^*(y, b')$ must coincide with $\hat{q}(y, b')$. Hence, $q^*(y, b')$ must be increasing in b'.

$\square$

Proposition 3 assumed that the bond decision rule conditional on repayment was increasing in b and showed that the pricing function is increasing in b'. The next Proposition shows that if the pricing function is increasing in b' then the bond decision rule is increasing in b. Thus Proposition 3 and 4 are "dual" of each other.

Proposition 4: If $q(y, b')$ is increasing in b' then in the region where repayment is feasible $a(y, m, b)$ is increasing in b.

Proof: Fix m and y and suppose that $b^1 < b^0$.

Denote $a(y, m, b^0)$ by ${b'}^0$ and the associated consumption level by c^0. Let $\hat{b}'$ be some other feasible choice greater than ${b'}^0$ and let $\hat{c}$ be the associated consumption level. Then, by optimality, we have

$$
\begin{aligned}
& u(c^0 - m) + \beta E_{(y'\, m')|y} \max\left\{V(y', m', {b'}^0), W(y', \bar{m})\right\} \\
& \geq \\
& u(\hat{c} - m) + \beta E_{(y'\, m')|y} \max\left\{V(y', m', \hat{b}'), W(y', \bar{m})\right\}
\end{aligned} \tag{6}
$$

Since $\hat{b}' > {b'}^0$, the fact that $V(y, m, b)$ is strictly increasing in b (Proposition 1) implies

$$
E_{(y'\, m')|y} \max\left\{V(y', m', \hat{b}'), W(y', \bar{m})\right\} > E_{(y'\, m')|y} \max\left\{V(y', m', {b'}^0), W(y', \bar{m})\right\}.
$$

Hence (6) implies $c^0 > \hat{c}$. Let $\Delta = c^0 - \hat{c} > 0$ denote the loss in current consumption from choosing $\hat{b}'$ over ${b'}^0$ when the beginning of period borrowings are b^0. Then from the budget constraint we have

$$
-q(y, {b'}^0)[{b'}^0 - [1-\lambda]b^0] - \Delta = -q(y, \hat{b}')[\hat{b}' - [1-\lambda]b^0],
$$

or,

$$
-q(y, {b'}^0){b'}^0 - \Delta = -q(y, \hat{b}')\hat{b}' + [q(y, \hat{b}') - q(y, {b'}^0)][1-\lambda]b^0.
$$

Observe that since $\hat{b}' > {b'}^0$ and, by hypothesis, $q(y, b')$ is increasing in b', the term $[q(y, \hat{b}') - q(y, {b'}^0)] \geq 0$.

Holding fixed $\hat{b}'$ and ${b'}^0$, let $\Delta(b^1)$ be the value of Δ that solves:

$$
-q(y, {b'}^0){b'}^0 - \Delta(b^1) = -q(y, \hat{b}')\hat{b}' + [q(y, \hat{b}') - q(y, {b'}^0)][1-\lambda]b^1.
$$

Then $\Delta(b^1)$ is the change in current consumption from choosing $\hat{b}'$ over ${b'}^0$ when beginning of period of asset level is b^1. Since $[q(y, \hat{b}') - q(y, {b'}^0)] \geq 0$, $b^1 < b^0$ implies $\Delta(b^1) \geq \Delta$. Thus the loss in current consumption from choosing $\hat{b}'$ over ${b'}^0$ is at least as large when the beginning of period asset holdings is b^1 as compared to b^0. Next, note that since $y - m + [\lambda + [1-\lambda]z]b^1 < y - m + [\lambda + [1-\lambda]z]b^0$, consumption under the choice of ${b'}^0$ when beginning of period of bond holdings is b^1, denoted c^1, is strictly less than c^0. It follows from strict concavity of $u(\cdot)$ that

$$u(c^1 - m) - u(c^1 - \Delta(b^1) - m) > u(c^0 - m) - u(c^0 - \Delta - m).$$

Therefore,

$$\begin{aligned}
&u(c^1 - m) - u(c^1 - \Delta(b^1) - m) \\
&> \\
&\beta E_{(y' \, m')|y} \max\left\{V(y', m', \hat{b}'), W(y', m')\right\} - \beta E_{(y' \, m')|y} \max\left\{V(y', m', {b'}^0), W(y', m')\right\}.
\end{aligned}$$

Since $\hat{b}'$ is any asset choice greater than ${b'}^0$ the optimal choice of b' (under repayment) when beginning-of-period bond holdings is b^1 cannot be greater than ${b'}^0$. Therefore, $a(y, m, b^1) \leq a(y, m, b^0)$.

□

Proposition 5: In the region where repayment is feasible $a(y, m, b)$ is decreasing in m.

Proof: Fix y and b and let $m^0 < m^1$. Denote $a(y, m^0, b)$ by ${b'}^0$ and the associated consumption by c^0. Let $\hat{b}' > {b'}^0$ be some other feasible choice of b' greater than $\hat{b}'^0$ and denote the associated consumption by $\hat{c}$. Then, by optimality,

$$\begin{aligned}
&u\left(c^0 - m^0)\right) + \beta E_{(y' \, m')|y} \max\left\{V(y', m', {b'}^0), W(y', \bar{m})\right\} \\
&\geq \\
&u\left(\hat{c} - m^0)\right) + \beta E_{(y' \, m')|y} \max\left\{V(y', m', \hat{b}'), W(y', \bar{m})\right\}
\end{aligned}$$

Since $V(y, m, b)$ is strictly increasing in b (Proposition 1), the above inequality implies $c^0 > \hat{c}$ (the implied inequality is strict as long as probability of repayment of ${b'}^0$ is strictly positive). Since $u(c - m^1) = u(c - m^0) - \int_{m^0}^{m^1} u'(c - x)dx$, we have that $u(c^0 - m^1) - u(\hat{c} - m^1) = u(c^0 - m^0) - u(\hat{c} - m^0) - \int_{m^0}^{m^1} \left[u'^0 - x) - u'(\hat{c} - x)dx\right]$. Since $c^0 > \hat{c}$, it follows from the strict monotonicity and concavity of $u(\cdot)$ (diminishing marginal utility) that $u(c^0 - m^1) - u(\hat{c} - m^1) > u(c^0 - m^0) - u(\hat{c} - m^0)$. Therefore ${b'}^0$ strictly dominates $\hat{b}'$ when minimum consumption requirement is m^1. Since $\hat{b}'$ was any asset choice greater than ${b'}^0$, it follows that $a(y, m^1, b)$ cannot exceed ${b'}^0$. Hence $a(y, m^1, b) \leq a(y, m^0, b)$.

□

The next proposition relates the set of m values for which there is default. Let $D(y, b) = \{m \in M : d(y, m, b) = 1\}$. Then we have:

Proposition 6: $D(y, b)$ is either the empty set, the whole interval M or a semi-open interval $(m^*, \bar{m}]$ where $m^* \in [0, \bar{m})$.

Proof: 3 cases are possible. (i) $V(y, 0, b) < W(y, \bar{m})$. Since V is strictly decreasing in m (Proposition 1), it follows that $V(y, m, b) < W(y, \bar{m})$ for all $m \in M$. Therefore, there will be default for every realization of m. In this case the default interval is the whole interval $[0, \bar{m}] = M$ (ii) $V(y, m, b) < W(y, \bar{m}) \leq V(y, 0, b)$. Then, by the continuity and strict monotonicity of V with respect to m, there exists a unique $m^* \in [0, \bar{m})$ such that $V(y, m^*, b) = W(y, \bar{m})$. Then the default interval is $(m^*, \bar{m}]$ (iii) $W(y, \bar{m}) \leq V(y, \bar{m}, b)$. In this case, $V(y, m, b)$ is at least as large as $W(y, \bar{m})$ for every realization of m and there will never be any default. Then, the default set is the empty set.

□

The final proposition concerns the existence of an equilibrium with the property that the equilibrium price function $q(y, b')$ is increasing in b'.

Proposition 7: There exists an equilibrium price function $q^*(y, b')$ that is increasing in b'.

Proof: (sketch) Let $\bar{q} = [\lambda + [1 - \lambda]z]/[\lambda + r_f]$. Then $\bar{q}$ is the present discounted value of a bond with coupon payment z and probability of maturity λ on which there is no risk of default. Let Q be the set of all non-negative functions $q(y, b')$ defined on $B \times Y$ that are increasing in b' and bounded above by $\bar{q}$. For any given $q \in Q$, Proposition 4 implies that $a(y, m, b; q)$ is increasing in b. And, given $a(y, m, b; q)$, $d(y, m, b; q)$ and q, by Proposition 3, $T(q)(y, b')$ is increasing in b'. Furthermore, $T(q) \leq \bar{q}$. Therefore, there is a mapping $F : Q \to Q$ whose fixed point is an equilibrium. Clearly Q is a compact convex set. We can also show that the mapping F is continuous. Therefore, by the Brower's Fixed Point Theorem, there exists $q^* \in Q$ such that $F(q^*) = q^*$.

□

6 Computation

It turns out that computing the equilibrium price function for bonds with maturity greater than one period is challenging. In this section we discuss the nature of the challenge and how this challenge is met in our paper.

To understand the new computational issues introduced by long-maturity bonds, it is useful to begin with the case of one-period bonds and no preference shocks. In this case, the equilibrium pricing function is computed via the following iteration

$$q^{k+1}(y,b') = E_{y'|y}\left[[1 - d(y',b';q^k)]\frac{1}{1+r_f}\right]. \tag{7}$$

Here q^k denotes the k-th iterate of the price function and $d(\cdot,\cdot,q^k)$ is the optimal default function given the price function q^k. Since there are no preference shocks, the state variables in this decision rule are simply endowments and beginning-of-period bond-holdings. For the iteration to converge, it is important that small changes between the k-th and the $k+1$-st iterate of the price function not imply a large change between the $k+1$-st and the $k+2$-nd iterate. However, because default is a discrete choice, a small change in the price function can lead to a switch in behavior from default to repayment (or vice versa). This will happen if for some q^k the sovereign is very close to indifference between the choice of default and repayment for some (y,b) pair. If a switch happens for a small change in the pricing function, the expectation in (7) is may change discretely. The reason for this is that the number of grid points on y is typically not large in applications and so each point on the y grid has signficant probability mass. Thus a discrete change in behavior for some y can change the expectation discretely. Observe that for any given y indifference between default and repayment will occur for very *specific* values of b. This means that the problem associated with indifference and switching is more likely to arise for a *fine* set of grid points on b because a fine set is more likely to contain a grid point at which there is near-indiffernce. In this sense, there is a trade-off between an accurate solution to the sovereign's decision problem and the ease with which the equilibrium pricing function can be computed.

This difficulty is compounded when bonds can last more than one period. In this case, ignoring preference shocks, the pricing equation is solved by iterating on

$$q^{k+1}(y,b') = E_{y'|y}\left[[1-d(y',b';q^k)]\frac{\lambda+[1-\lambda][z+q^k(y',a(y',b';q^k))]}{1+r_f}\right] \tag{8}$$

Now, the the calculation of the $k+1$-st iterate depends on the $a(\cdot,\cdot;q^k)$ decision rule as well. This creates two problems. First, it is no longer possible to have a coarse grid on bond holdings. A coarse grid would imply that whenever a small change in the pricing function induces the sovereign to switch her desired level of bond-holdings, the switch would affect the expectation in (8) discretely. But making the grid finer does not overcome this problem because of the possible non-convex nature of the optimization problem. This is so because the budget set under repayment may not be convex (because $q(y,b')$ is a non-linear function of b') or because future expected utility (as a function of b') has non-concave segments (see Figures 1 and 2). These non-convexities imply that, given (y,b), the sovereign may be indifferent between two widely separated values of b'. Thus there may be jumps in the decision rule $a(y,b;q^k)$ viewed as a function of q^k. These jumps in decisions then cause large changes in the future value of the outstanding bonds and therefore in the current price.

To summarize, the jumps (or discontinuities) in the decision rules stem from the possibility of default and the resulting non-convexity of the budget set under repayment and non-concavity of the value function. Thus the jumps are an intrinsic part of the decision problem being studied here and cannot be avoided. Given this, the only approach to solving the problem is to arrange matters so that the jumps do not affect the expected value in (8) too much. One way to ensure this is to make each y value occur with a very low probability – i.e., increase the grid size on y. This, in effect, is the strategy employed in Chatterjee *et al* (2007) and in this paper as well. What we do is introduce another exogenous stochastic state variable, in our case the minimum consumption requirement m, which is drawn from a *continuous* distribution. Thus the state variable for the sovereign becomes the triplet (y,m,b). Now, the jumps in behavior (switching between default and repayment or switching between widely separated values of debt) occur at specific values of (y,m,b). Since m is drawn from a continuous distribution, these values occur with negligible probability. Therefore, if a small change in the q function induces a jump in behavior at one of these points, the expected value in the analog of (8) is not affected by the jump.

However, introducing an additional *continuous* state variable m comes with its own set of complications. In order to compute the expected value in the pricing equation we need to be able to locate the values of m at which there is a switch in behavior. For instance, we need to be able to locate the value of m at which the sovereign switches from repayment to default or vice versa.

This is where the characterization of behavior with respect to m, namely, Propositions 5 and 6, come in useful. For instance, Proposition 6 indicates that we need to locate one value of m at which the sovereign is indifferent between repayment and default, if such a point exists. Proposition 5 indicates that the choice of b' is decreasing in m – as m increases the optimal decision switches to a lower value of b'. It is important to note that both results relied on the fact that m was an iid shock. In contrast, making y continuous and searching for thresholds on y is considerably harder because y is persistent and affects future expected values as well.

However, Proposition 5 does *not* imply that as m increases the next optimal choice of b' will be *adjacent* to the current optimal choice. As noted earlier, the non-linearity of the pricing function or non-concavity of the value function may imply that the sovereign finds it optimal to switch between widely separated values of b'. This fact raises a challenge in the computation. Evidently, Proposition 5 implies that there exists $\{m^1 < m^2 < \ldots < m^{K-1} < \bar{m}\}$ and $\{b'^1 > b'^2 > \ldots > b'^K\}$ such that b'^1 is chosen for all $m \in [0, m^1)$, b'^2 is chosen for all $m \in [m^1, m^2)$, $\ldots$, b'^K is chosen for all $m \in (m^{K-1}, \bar{m}]$. But, since b'^{k-1} need not be adjacent to b'^k, how does one find the values of b'^k and the associated values of m^k?

To understand the algorithm we use to determine the $\{(m^k, b'^k)\}$ pairs, imagine that we have located the pairs $\{(m^1, b'^1), (m^2, b'^2), \ldots (m^g, b'^g)\}$. To proceed, compare the utility from choosing b'^g with the utility from choosing the next lower asset level b'^- (higher debt level) for different values of m. Two cases are possible.

1. b'^g is better than b'^- for all $m \in M$. This will happen if borrowing more does not increase consumption today. Then, we drop b'^- from further consideration and move to comparing b'^g to the next lower asset level.

2. There is a value of m, denoted $\tilde{m}$ for which the two choices give the same utility. Here two cases are possible:

   (a) If $\tilde{m} \geq m^g$ then we add $(\tilde{m}, b'^-)$ to the list of pairs and proceed to the compare the utility between b'^- with the next lower asset level.

   (b) If $\tilde{m} < m^g$, we drop b'^g from further consideration and proceed backwards to compare b'^- with b'^{g-1}. The reason is that $\tilde{m} < m^g$ implies that b'^- is preferred to b'^g for any $m > \tilde{m}$ and at the same time b'^{g-1} is preferred to b'^g for any $m < m^g$. This implies that b'^g is dominated by the choices of b'^{g-1} and b'^- and will never be chosen. Thus it can be dropped from further consideration. But if this is the case then b'^- needs to be compared to b'^{g-1}. The process is continued by finding a new $\widetilde{m}_2$ between the choices of b'^- and b'^{g-1}. If $\tilde{m}_2 \geq m^{g-1}$, then we add $(\tilde{m}_2, b'^-)$ to the list of pairs and proceed to compare the utility between b'^- with the next lower level of asset. If $\tilde{m}_2 < m^{g-1}$, we drop b'^{g-1} from further consideration, and continue to go backwards through the list. This process will either end in finding m^{g-j} that is less than or equal to $\tilde{m}_2$ or in the exhaustion of all pairs in the list $\{m^k, b'^k\}$. If the latter we conclude that b'^- dominates any $b' > b'^-$ for all m and proceed to compare b'^- with the next lower level of debt.

To implement this algorithm we start off with the list $\{(\bar{m}, 0)\}$ (meaning that no borrowing is optimal for all m) and then proceed to compare 0 with the next level of debt. The algorithm is applied until every element of B has been picked up and compared to the existing list.

7 Maturity, Indebtedness and Spreads: The Argentine Case

We apply the framework developed in the previous sections to the Argentine case. Our objective is to simultaneously account for the average default spreads on Argentine bonds as well as the average indebtedness of Argentina over the 10-year period between 1993:Q1 and 2001:Q4. This is also the time period analyzed in Arellano (2007). Arellano focused on understanding the average default spreads on Argentine bonds but did not attempt to match Argentina's average debt level. The main contribution of our quantitative work is to establish that allowing for long-duration bonds, besides being a closer fit with reality, helps to match both the level of spreads and the level of average indebtedness.

For the quantitative work we make the following specific functional form or distributional assumptions.

- Endowment process: The stochastic evolution of y_t is governed by an AR-1 process in logs:

$$\ln y_t = \rho \ln y_{t-1} + \epsilon_t, \text{ where } 0 < \rho < 1 \text{ and } \epsilon_t \text{ distributed } N\left(0, \sigma_\epsilon^2\right).$$

- Preference shock process: The preference shock m is drawn from a truncated normal distribution with support $[0, \bar{m}]$ centered at $\bar{m}/2$ and with variance σ_m^2 .
- Utility function: The utility function is assumed to be the CRRA form $(c-m)^{1-\gamma}/(1-\gamma)$
- Following Arellano (2007), the output loss in the event of default or exclusion is assumed to be of the form:

$$\phi(y) = \begin{cases} 0 & \text{if } y \leq \overline{y} \\ y - \overline{y} & \text{if } y > \overline{y} \end{cases}.$$

With these assumptions, the numerical specification of the model requires giving values to 10 parameters. These are (i) four preference parameters β, γ, $\bar{m}$ and σ_m^2, (ii) two endowment process parameters ρ and σ_ϵ^2, (iii) two parameters describing the bond, the maturity parameter λ and the coupon payment z, (iv) the default output loss parameter $\bar{y}$, (v) the probability of re-entry following default, ξ, and (vi) the risk-free rate r_f.

The parameter selections proceeds as follows. Of the preference parameters, the value of the γ is set equal to 2, which is the standard value used in this literature. The precise value of $\bar{m}$ is not important to the results and so it was set at a relatively small number 0.046. The endowment process is determined by fitting the AR-1 process to the quarterly real GDP data. The estimation yields a value of $\rho = 0.9347$ and $\sigma_\epsilon = 0.0263$. The parameters describing the bond is determined to match the maturity and coupon information for Argentina reported in Broner, Lorenzoni and Schumkler (2007). The average coupon rate is about 12 percent per annum, or 0.03 per quarter, and the median maturity of Argentine bonds is 5 years or 20 quarters. Thus, $z = 0.03$ and $\lambda = 1/20 = 0.05$. Re-entry into the financial market following default usually occurs between 2 to 3 years and so $\xi = 0.10$ which gives average period of exclusion of 10 quarters or 2.5 years. The risk-free rate, r_f was set at 0.01, which is roughly the real rate of return on 3-month (one quarter) US Treasury bill.

The three remaining parameters β, $\bar{y}$, σ_m^2, are picked to match as closely as possible 70 percent of the average debt-to-GDP ratio, the average default spreads and the standard deviation of consumption expenditures relative to the standard deviation of GDP. We seek to match only a portion of the Argentine debt because we do not model repayment. In reality, most sovereign debt that goes into default pays off something. In Argentina's case, the repayment on defaulted debt has been around 30 cents to the dollar. Thus, we treat only 70 cents out of each dollar of debt as the truly unsecured portion of the debt.

We need to determine exactly what in the model corresponds to the observed debt-to-GDP ratio and the observed default spreads. In the data, the value of a bond is a weighted average of its market value at time of issue and the face value of the bond, where the weight on latter is higher closer is the bond to maturity. The value of total debt is just the sum of the value of individual bonds calculated in this way. Since the bonds in our model do not have a maturity date, we cannot replicate this procedure exactly. What we do instead is value the promised stream of payments on a bond at the risk-free rate so that we can capture the averaging between the market value at the date of issue and the face value.

The default spread in the model is calculated as in the data. Namely, given the unit price $q_{\lambda,z}(y, b)$ of the outstanding bonds, we calculate an internal rate of return that makes the present discounted value of the promised sequence of future payments on a unit bond equal to the unit price. If the interest rate is r, the present discounted value of the promised sequence of payments on a (λ, z) bond is $[\lambda + (1 - \lambda)z]/[\lambda + r]$. The required internal rate of return is given by $r(y, b)$ such that $q_{\lambda,\, z}(y, b') = [\lambda + (1 - \lambda)z]/[\lambda + r(y, b)]$. The difference between $r(y, b)$ and r_f is the default spread.

The parameter selections is summarized in the following two tables. Table 1 lists the values of the parameters that are selected directly without solving for the equilibrium of the model. Table 2 lists the parameter values that are selected by solving the equilibrium of the model and choosing the parameters so as to make the model moments come as close as possible to the data moments.

Table 1

Parameter	Description	Value
γ	risk aversion	2
$\bar{m}$	upper bound on m	0.046
σ_ϵ	standard deviation of ϵ	0.0263
ρ	autocorrelation	0.9347
ξ	probability of reentry	0.10
r_f	risk free return	0.01
λ	reciprocal of ave. maturity	0.05
z	coupon payments	0.03

Table 2

Parameter	Description	Value
σ_m	standard deviation of m	0.009
β	discount factor	0.969
$\bar{y}$	default punishment	0.881

The results are reported in Table 3. The top row reports the data for Argentina. The average risk spread over this period, as calculated from EMBI data, is around 8.8 percent. The average debt-to-GDP ratio is 1.00. The debt service ratio, which is the ratio of the average annual payment on debt (interest and principal) to GDP is 5.5 percent. And, finally, the ratio of the standard deviation of consumption expenditures to the standard deviation of GDP is around 110 percent.

Table 3

	Def. Freq.	Spd	D-to-Y	D Srvc	σ_C/σ_Y
Data		0.0876	1.00	0.055	1.104
$\lambda = 0.05, z = 0.03$	0.0594	0.0876	0.70	0.041	1.101
Arellano ($\lambda = 1$)	0.0300	0.0358	0.06	0.053	1.098
$\lambda = 1$	0.0034	0.0038	0.47	0.469	1.153

The second row reports the same moments in the model.[2] Recall, that we search over the values of β, $\bar{y}$ and σ_m^2 to match the average spreads, 70 percent of the debt-to-GDP ratio and the relative volatilities of consumption and GDP. It is evident that the matching exercise is quite successful. We do not target the debt service ratio but the agreement between model and data is fairly good for this statistic as well. We also report the default frequency in the model, which turns out to be little under 6.00 percent. This statistic is calculated conditional on the country being in debt and in good standing. We do not report the appropriate data moment for this statistic because we simply do not observe sufficiently many defaults in the data (for any one country) to make the data meaningful.

The third row of the Table reports, for comparison, the statistics as given in Arellano (2007) for the variable in question. Notice that the model moments are badly off for the debt-to-income ratio. Thus, clearly, comparing the second row to the third row, adding long-term debt is an improvement.

Although second moments are not the focus of this paper, we examine the cyclical properties of the economy. Figure 3 shows the simulated path of spreads in the model compared to the data and the simulated path of spreads in Arellano (2007). Notice that the model-simulated spread hugs actual spreads pretty closely. In particular, it manages to capture the rise in spreads toward the end of the sample period rather well.

Table 4 reports the standard deviations and correlations between main variables of interest. Observe that the model has the feature that the trade balance (NX) and the default spreads are countercyclical. Also, the spread and trade balance are positively correlated: when the spreads are high, borrowing is low (or, equivalently, NX is high).

Table 4 (S.D. and Correlations)

[2] In calculating these statistics we exclude the first 20 periods following re-entry into the international capital markets. In the period following re-entry, the debt and spread statistics are atypical in the model because the country has no debt and begins to accumulate debt pretty much regardless of the level of output. This debt accumulation phase attenuates the negative correlation between NX and y if the correlation is calculated inclusive of the debt-accumulation phase. In reality, Argentina emerged from default with debt because it agreed to make some payments on debt outstanding prior to default. Thus it did not have the same incentive for rapid debt accumulation following re-entry as present in the model.

	y	c	spread	NX
y	0.0595	0.9815	−0.7139	−0.3570
c		0.0657	−0.7748	−0.5287
spread			0.0868	0.6142
NX				0.013

The close concordance between data and model moments in shown in Table 3 raises the question: to what extent is this concordance the result of including long-term debt? There are two ways of answering this question. One way is to simply ask what happens to the equilibrium of the model if all debt is restricted to be one-period. The results from this exercise are shown in the bottom row of Table 3. Observe that both the default premium (spreads) and the debt-to-income ratio is much lower. The average spread is now around 0.4 percent – essentially zero. The debt-to-income ratio drops to 47 percent. The debt service is very close to the debt-to-income ratio because all debt is due in one-period. Finally, the volatility of consumption rises as the sovereign engages less in consumption smoothing – the standard deviation of consumption is around 115 percent of the standard deviation of output.

A second way to answer this question is to ask what is the best that a model with one-period debt can do in terms of matching the debt and spread statistics? Tables 5-7 show the results of this exercise. The first five rows of Table 5 are exactly the same as the corresponding rows in Table 1. The last two rows are different: now, all debt is one-period and there are no coupon payments. Table 6 shows the value of the two parameters, β and $\bar{y}$, that are being chosen to match two data moments – average spreads and the debt-to-income ratio. For reasons that will become clear, we do no attempt to match the relative volatility of consumption and so set $\bar{m}$ and σ_m^2 to zero. Notice that the value of β is quite low. There is not much change in the value of $\bar{y}$.

Table 5

Parameter	Description	Value
γ	risk aversion	2
$\bar{m}$	upper bound on m	0
σ_ϵ	standard deviation of ϵ	0.0263
ρ	autocorrelation	0.9347
ξ	probability of reentry	0.10
r_f	risk free return	0.01
λ	reciprocal of ave. maturity	1
z	coupon payments	0

Table 6

Parameter	Description	Value
σ_m	standard deviation of m	0.000
β	discount factor	0.7156
$\bar{y}$	default punishment	0.8417

The first row of Table 7 reports the results. It turns out that it is possible to match the debt-to-income ratio and the average spreads pretty closely even in a model with one-period debt but this success comes at the cost of a huge increase in the volatility of consumption relative to output. Essentially, the model matches the debt and spread statistics by making the sovereign very impatient. Greater impatience generates more borrowing and higher spreads. But the volatility in spreads, because all debt must be "rolled over" each period, implies a much higher level of volatility in net exports and, therefore, consumption. Observe that consumption is close to being 190 percent more volatile than output. If we expanded the set of targets to include the relative volatility of consumption and insisted that the model output match the relative volatility closely, the results would resemble Arellano (2007). The model will match the relative volatility closely but not the average spread or the average debt-to-output ratio.

Table 7

	Def. Freq.	Spread	Debt-to-GDP	Debt Service	σ_C/σ_Y
Data		0.0876	1.00	0.055	1.104
$\lambda = 1, z = 0$	0.0737	0.0839	0.70	0.703	1.878
$\lambda = 0.05, z = 0.03$	0.0598	0.0873	0.70	0.041	1.103

To understand why long-term debt improves model performance we can look at Figures 4 and 5. These figures show how spreads behave with respect to borrowing in the models with long-term and short-term debt. Figure 4 shows these relationships when output is above trend. Observe that spreads are higher and rise more gradually for the long-term bond case than the short-term bond case. The spreads are higher because lenders anticipate that the country will borrow more in the future and thereby inflict capital losses on them. The spreads rise gradually because the incentive to default does not rise rapidly with debt – the fact that the country needs to "roll-over" only a portion of its debt makes repayment an attractive option. The same gradual increase in spreads is evident in Figure 5 as well which shows the situation for below-trend output (observe that the scale for spreads is very different).

We can obtain some further intuition on how the model works by considering the sovereign decision to issue additional debt. Although this choice is discrete in the model, we may think of it being continuous for the moment. Also, for the moment, ignore the preference shock. Then, the marginal gain from borrowing is given by:

$$\left(-q(y,b') - \frac{\partial q(y,b')}{\partial b'}\left[b' - (1-\lambda)b\right]\right) u'\left(y + \lambda b - q(y,b')\left[b' - (1-\lambda)b\right]\right)$$

To understand the expression, it is easier to think of the country as the "monopolistic" supplier of bonds. When the country issues an extra unit of bond, it gets revenue from that extra unit sold, but at the same time the decrease in the price of the bond (because of increased default probability) decreases the revenue on all bonds that are currently issued. The country borrows more with long-term bond because both the price schedule is flatter with long-term bond (as seen in the figures), and also because the decrease in price of the bond by extra borrowing only affects the currently issued bonds, and with long-term bonds, the currently issued bonds is just a small portion of the outstanding debt.

The average spreads are much higher with long-term bond. There are two reasons for this. One is, as we see in the above equation, the country more readily enters these high probability of default regions since it does not internalize the effect increasing the level of debt has on decreasing the value of bonds issued in previous periods (which explains why the lenders take into account that the country will increase its debt in upcoming periods and charges high spreads from the start). The second reason is that, with long-term bonds, just before defaulting, the country issues high levels of debt at very high spreads. Typically, with long-term bond, it is not very costly to not default one more period (by just paying the portion λ of the debt). But by borrowing at high spreads the country can disproportionately tax the high income outcomes next period (where it wont default), and consume more today. And it does borrow at very high spreads just before defaulting with long-term bond. In contrast, with short-term debt this opportunity never arises, since raising enough money at very high spreads to refinance all existing debt becomes impossible.

8 Conclusion

This paper developed a tractable model of long-term unsecured debt and applied it to understanding the behavior of debt and default spreads in an emerging market, namely, Argentina. We showed that introducing long-term debt improves upon existing quantitative models of sovereign debt in matching the average default spreads and the average level of the debt-to-output ratio.

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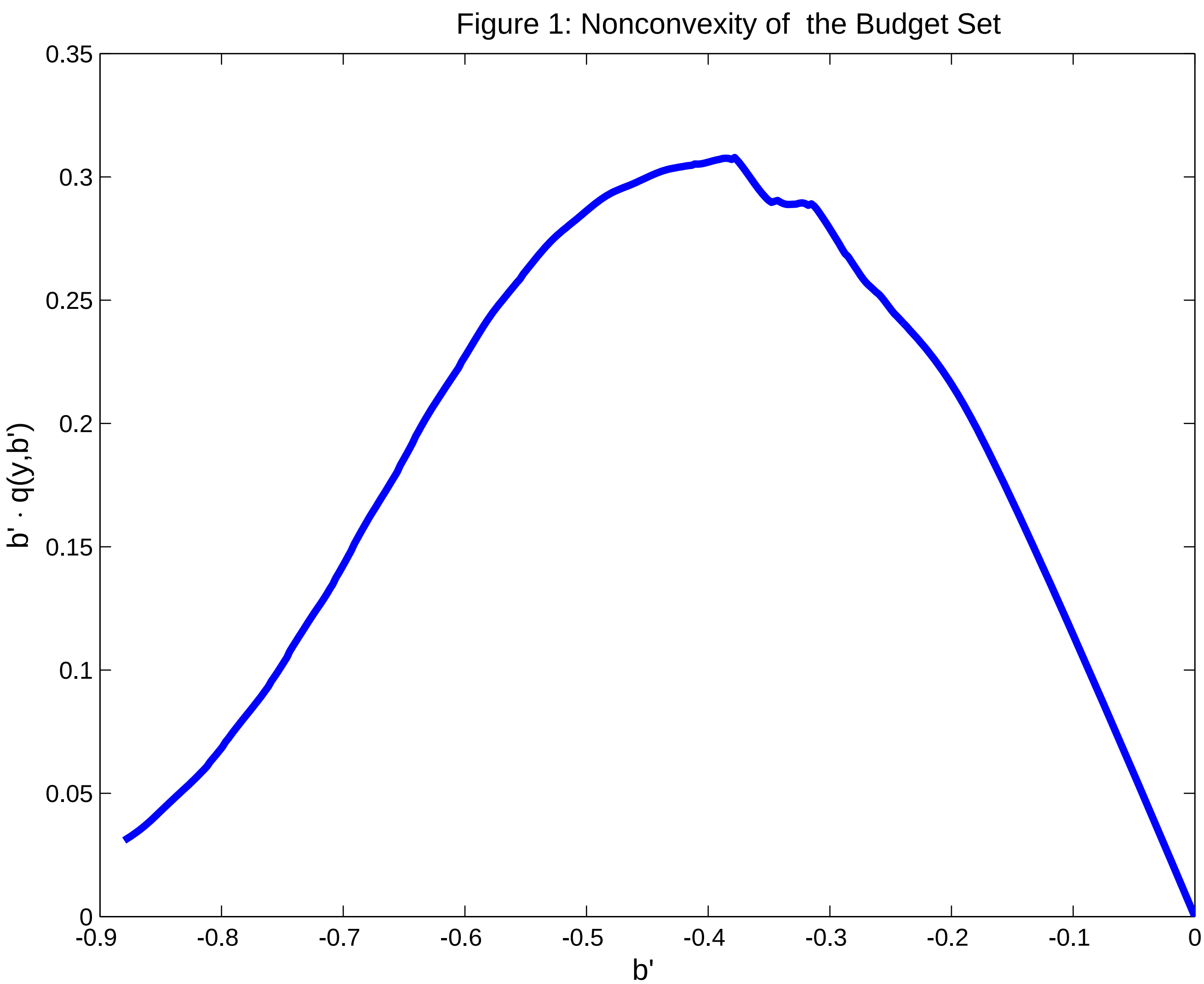
Figure 1: Nonconvexity of the Budget Set
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
b' · q(y,b')
-0.9
-0.8
-0.7
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1
0
b'

Figure 2: Nonconcavity of $E_{(y',m')|y}$ Max {V(y',m',b'), W(y', m)}

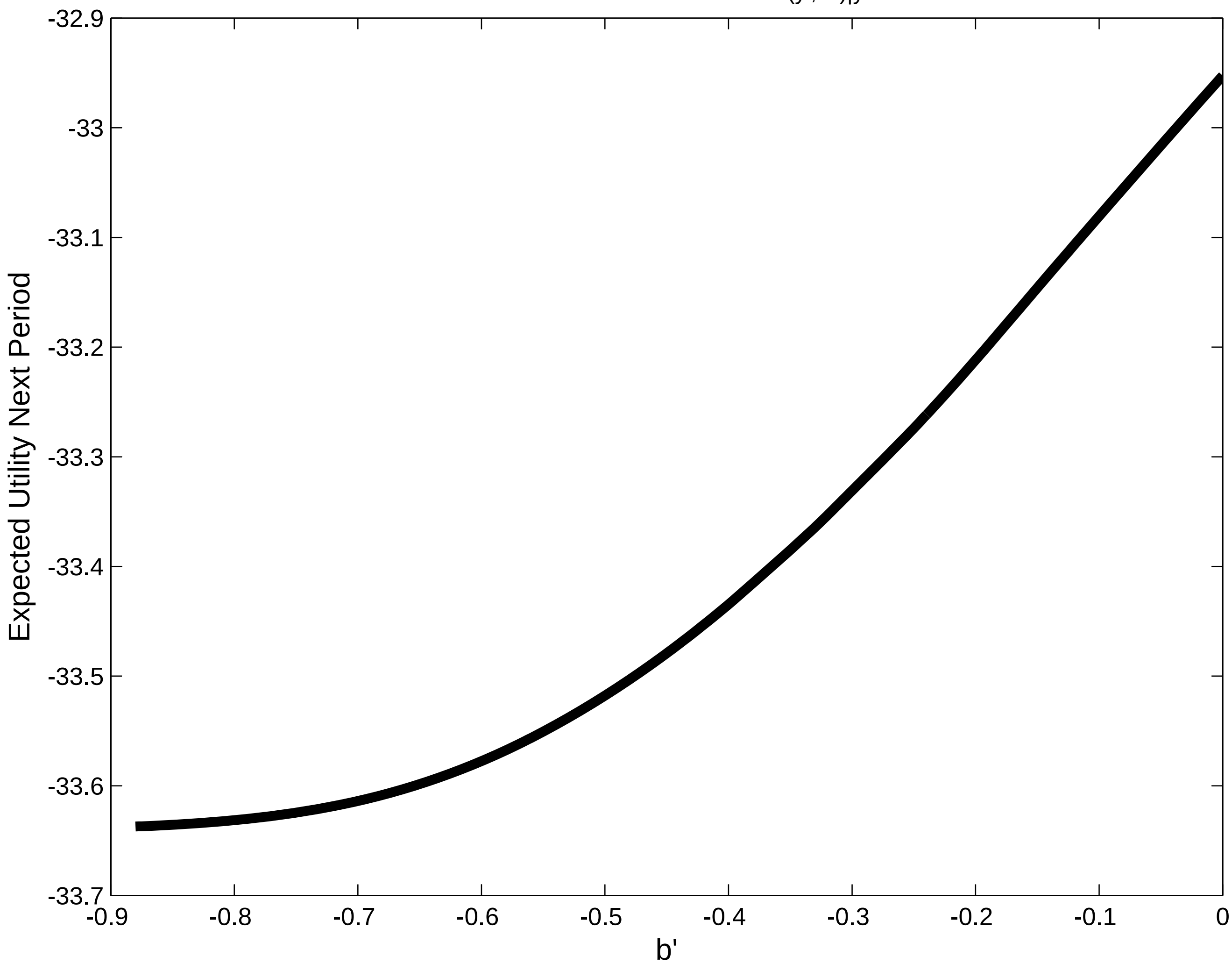

Figure 3: Simulated Spreads for Model and Data

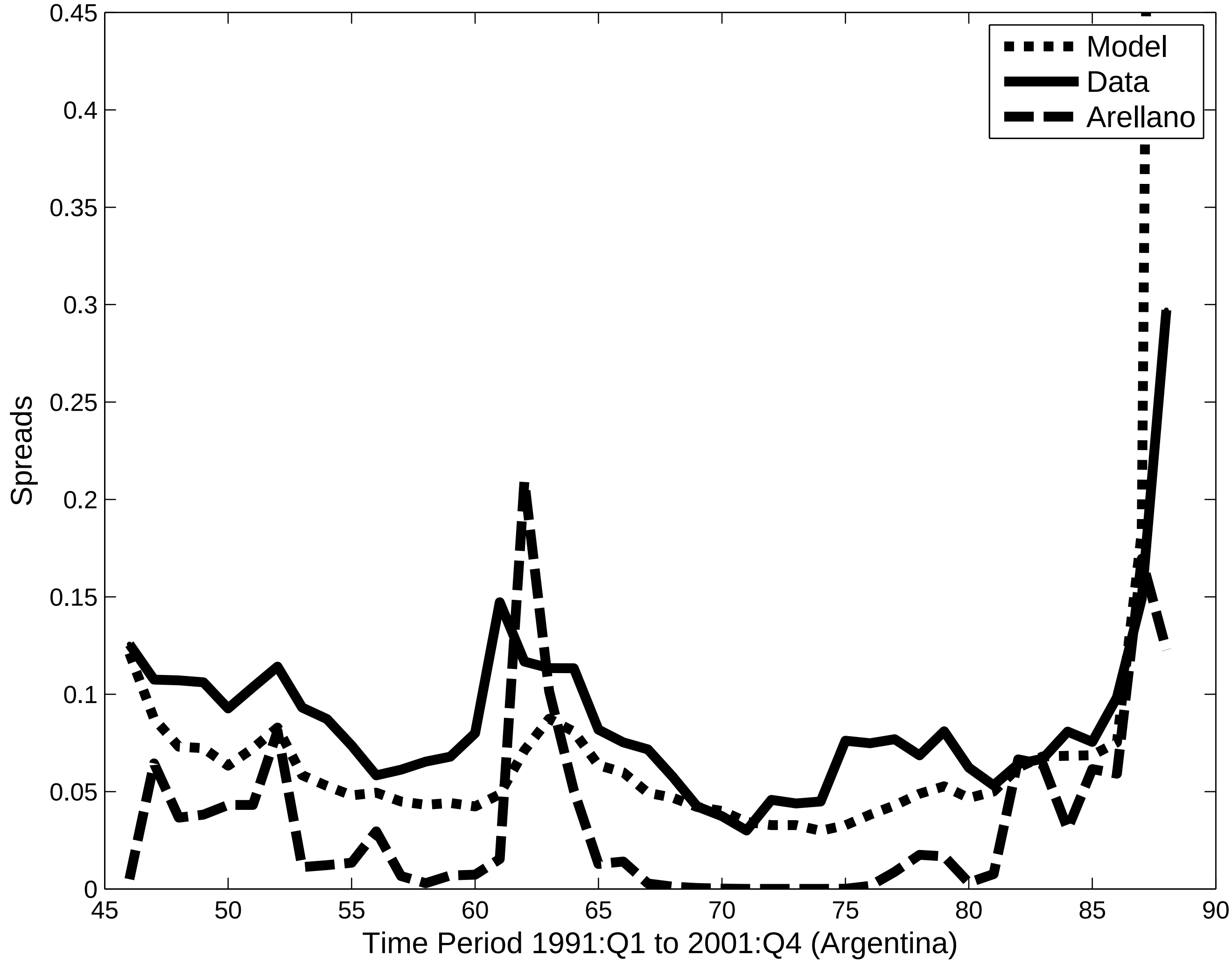

Figure 4: Spreads for Long-term and Short-term Bonds (Above-Trend y)

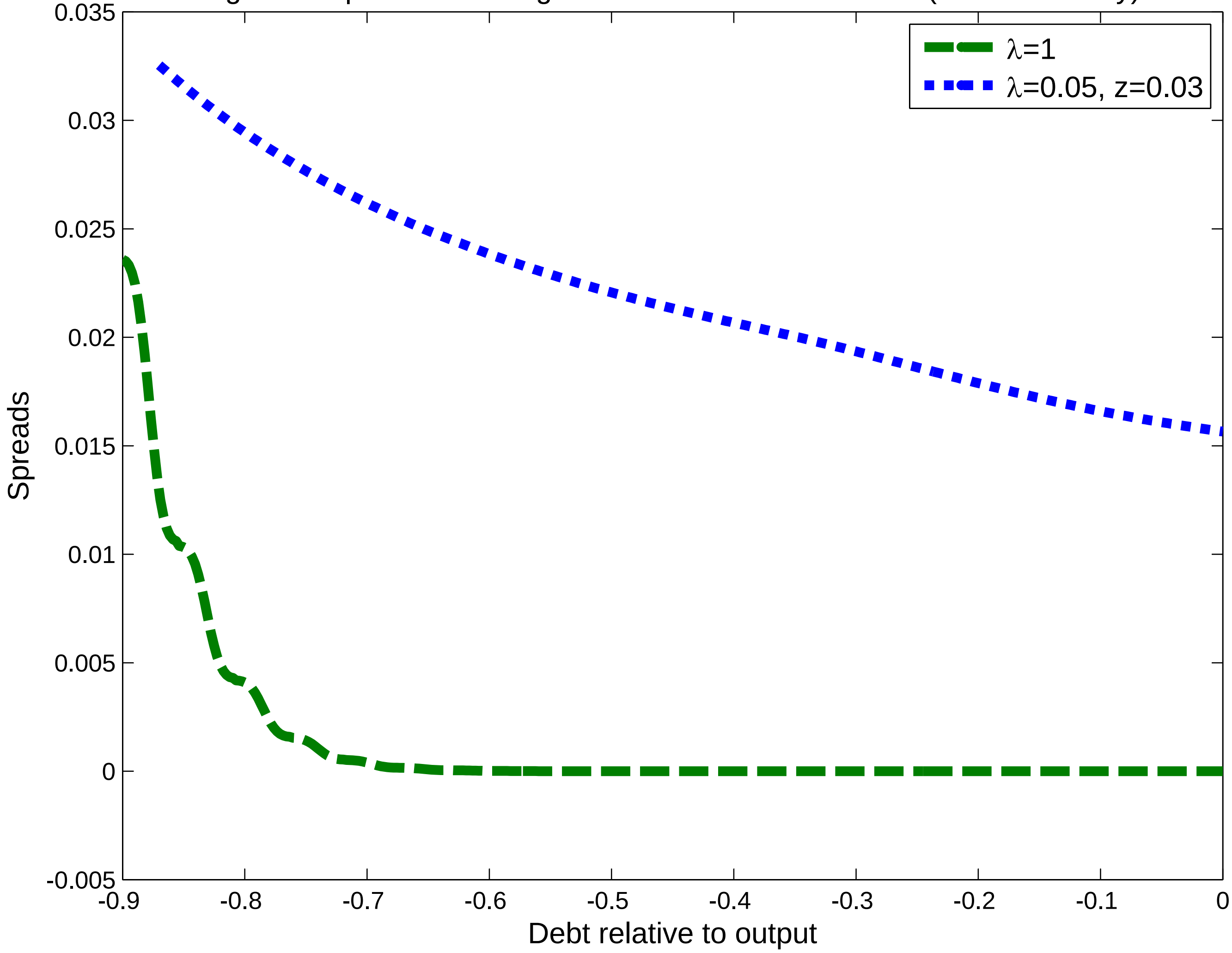

Figure 5: Spreads for Long-term and Short-term Bonds (Below-Trend y)