On the Investment Implications of Bankruptcy Laws*

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Abstract

Axiomatic analysis of bankruptcy problems reveals three major solution rules: (*i*) the proportional rule (PRO), (*ii*) the constrained equal awards rule (CEA), and (*iii*) the constrained equal losses rule (CEL). The literature offers quite a large number of characterizations for each one of these rules and in this sense, no one of the three seems to be more central than the others. However, most real life bankruptcy procedures implement only one of these rules: PRO. (The typical real life rule first partitions the claimants into priority groups, then uses PRO in each group). In this paper, we try to explain why in applications PRO has been more popular than the other three. One possible explanation, overlooked by the axiomatic literature, may lie in these rules' implications on the investment behavior in the society. To explore this possibility, we construct a noncooperative game in which two players simultaneously choose how much to invest in a risky firm. (The players have CARA utility functions on money with possibly different risk aversion parameters). The outcome of investment is uncertain: if the firm succeeds, each player receives a positive return on his investment; however, with a positive probability, the firm goes bankrupt. In case of bankruptcy, the assets of the firm are liquidated and allocated among the investors according to a prespecified bankruptcy rule. For each of the four bankruptcy rules, we analyze its induced Nash equilibrium investment behavior. Particularly, we compare the total equilibrium investment induced by PRO to that of the other three rules. Our first result is that the equilibrium total investment under PRO always exceeds that of CEA. The comparison of PRO to CEL is not as sharp. There are three types of equilibria of the investment game under CEL and two of them always induce higher total investment than does PRO. The third type of equilibrium under CEL, on the other hand (*i*) always induces less total investment than PRO and (*ii*) induces more investment than CEA if the agents are sufficiently different in their attitudes toward risk.

1 Introduction

Following the seminal work of O'Neill (1982), a vast literature focused on the axiomatic analysis of "bankruptcy problems". As the name suggests, a canonical example to this problem is the case of a bankrupt firm: the monetary worth of a bankrupt firm (hereafter, the *estate*) is to be allocated among its creditors. Each creditor holds a *claim* on the firm and the firm's liquidation value is less than the total of the creditors' claims.

The axiomatic literature provided a large variety of "bankruptcy rules" as solutions to this problem. Among them, three are significantly more prominent than the others. The *Proportional rule (PRO)* suggests to allocate the estate proportionally to the claims. The *Constrained Equal Awards rule (CEA)* suggests equal division of the estate, subject to the constraint that no agent receive more than his claim. The *Constrained Equal Losses rule (CEL)* alternatively suggests to equate across agents, the difference between his claim and his share, subject to the constraint that no agent receive a negative share.

There is also a more applied literature on real-life bankruptcy problems (e.g. see Atiyas, 1995; Hart, 1999, Stiglitz, 2001). While this literature focuses on a wide range of issues and cases, it also deals with the problem considered by the axiomatic bankruptcy literature.[1] It follows from this applied literature that, in real-life bankruptcy problems of this kind, almost every country uses the following rule. First, creditors are sorted into different priority groups. For example, government organizations, corporate creditors, and individual creditors might form different priority groups. These groups are served sequentially. That is, a creditor is not awarded a share until creditors in higher priority groups are fully reimbursed. Second, in each priority group, the shares of the creditors are determined in proportion to their claims, that is, according to PRO.

The actual bankruptcy laws seem to have preferred PRO over the other three rules. The rationale behind this choice might be purely historical. Or it might be that governments attach more importance to the axioms that characterize PRO over the axioms that characterize the other rules.

In this paper, we explore another explanation to the popularity of PRO. Alternative bankruptcy rules affect the investment behavior in a country in different ways. In a way, each rule induces a different noncooperative game among the investors. Comparing the equilibria

[1] In the US bankruptcy law, this problem is referred to as Chapter 7 bankruptcy.

of these games, in terms of total investment or social welfare, might provide us ways of comparing alternative bankruptcy rules in a way that is not considered in the axiomatic literature.

To this end, for each bankruptcy rule, we construct a simple game among a set of investors who simultaneously choose how much money to invest in a firm. The total of these investments determine the value of the firm. The firm is a lottery which either brings a positive return or goes bankrupt. In the latter case, its remaining value is allocated among the investors according to a prespecified bankruptcy rule. For each bankruptcy rule, we analyze the Nash equilibria of the corresponding investment game. We then compare these equilibria.

In our model, agents have Constant Absolute Risk Aversion preferences and are ordered according to their degree of risk aversion. The agents do not face liquidity constraints and thus, their income levels are not relevant. However, as is standard in the literature, it is possible to interpret an agent's degree of risk aversion to be negatively correlated by his income level. Namely, richer agents are less risk averse than poorer ones. Alternatively, each agent can be taken to be an investment fund. In this case, the income level is irrelevant. The risk-aversion parameter attached to each investment fund can be interpreted as the type of that fund.

Our main results are as follows. In Section 3, we analyze Nash equilibria under PRO, CEA, abd CEL respectively. In Proposition 1, we show that there always is a unique dominant strategy equilibrium under PRO. For the other three rules, we present examples to demonstrate that a Nash equilibrium does not always exist. Thus, the other results, Propositions 3 (for CEA), 4 (for CEL), 15, and 18 (both for TAL), present necessary conditions on equilibria. In Proposition 3, we show that there is only one type of Nash equilibrium in which no agent is fully refunded and thus, they receive equal shares in case of bankruptcy. This is not the case for the CEL rule. In Proposition 4, we show that three different types of equilibria are possible under the CEL rule. In a *Type 1 equilibrium under CEL*, both agents receive positive shares in case of bankruptcy (and thus choose positive investment levels). In a *Type 2 equilibrium under CEL*, one agent chooses zero investment. The other agent, being the only investor, receives a share proportional to his investment (and thus, his optimal investment level is equal to that under PRO). In a *Type 3 equilibrium under CEL*, one agent receives zero share in case of bankruptcy but nevertheless, chooses a positive investment level.

We then compare these equilibria in Section 4. In Proposition 5, we show that in equilibrium, PRO induces more total investment than CEA. We then compare equilibria under CEL and PRO. In Propositions 6 and 8, we show that Type 1 and 3 equilibria under CEL induce more total investment than PRO. We then show in Proposition 7 that Type 2 equilibria under CEL induces less total investment than PRO. This is because in a Type 2 equilibrium one agent chooses zero investment and the other chooses the same investment level as he would choose under PRO.

Finally, we compare in Proposition 9, total investment under the CEL and CEA rules. Due to previous results, total investment in a Type 1 or Type 3 equilibrium, under the CEL rule exceeds total investment under the CEA rule. A Type 2 equilibrium can however induce smaller total investment if the agents are sufficiently similar in terms of their risk aversion attitudes.

Related Literature.

The axiomatic literature on bankruptcy and taxation problems contains many studies that analyze the properties of one or several of the four main rules. For example, Dagan (1996), Schummer and Thomson (1997), Herrero and Villar (2002), and Yeh (2001) analyze properties of CEA. Yeh (2001a), Herrero and Villar (2002), and Herrero (2003) analyze properties of CEL. Aumann and Maschler (1985) and the following literature discusses properties of TAL. O'Neill (1982), Moulin (1985a,b), Young (1988), Chun (1988a), de Frutos (1999), Ching and Kakkar (2000), Chambers and Thomson (2002), and Ju, Miyagawa, and Sakai (2007) analyze properties of PRO. Thomson (2003 and 2008) presents a detailed review of the extensive axiomatic literature on bankruptcy and taxation problems.

There are previous papers that employ game theoretical rules to analyze bankruptcy problems. Aumann and Maschler (1985), Curiel, Maschler, and Tijs, (1987), and Dagan and Volij (1993) relate bankruptcy rules to cooperative game theoretical solutions. Chun (1989) presents a noncooperative game that implements an egalitarian surplus sharing rule. Dagan, Serrano, and Volij (1997) present a noncooperative game that implements a large family of consistent bankruptcy rules by employing the rule's two-person version in the design. Chang and Hu (2008) carry out a similar analysis for a class of "f-just" rules. Herrero (2003) implements the Constrained Equal Awards and Constrained Equal Losses bankruptcy rules. Garcia-Jurado, Gonzalez-Diaz, and Villar (2006) present noncooperative games for a large class of “acceptable” rules. None of these paper however focus on investment implications of bankruptcy rules.

This paper is closely related to Karagozoglu (2008) who also designs a noncooperative game and analyzes investment implications of a class of bankruptcy rules that contains PRO, CEA, and CEL. The main differences between our models are as follows. First, in Karagozoglu (2008) there are two types of agents (high income and low income) and there is an arbitrary number of agents of each type. Due to complexities introduced by the following items, we focus on the two-agent case. Second, in Karagozoglu (2008) agents choose between zero or full investment of their income. In our model, agents are allowed choose any amount of investment. Third, in Karagozoglu (2008) agents are risk neutral. In our model, agents are risk averse and possibly heterogenous in their attitude towards risk. (Alternatively, Karagozoglu (2008) introduces heterogeneity in income levels). Our results are in agreement in comparison of total investment under the PRO and CEA rules. Due to the aforementioned differences, however, we obtain different results when comparing the PRO and CEL rules. In Karagozoglu (2008) total investment induced by PRO exceeds that of CEL. In our model, however, this only happens in one type of equilibrium. There are two other types of equilibria where CEL induces more total investment than PRO.

2 Model

Let $N = \{1, 2\}$ be the set of agents. Each $i \in N$ has the following **Constant Absolute Risk Aversion (CARA) utility function** $u_i : \mathbb{R}_+ \to \mathbb{R}$ on money:

$$u_i(x) = -e^{-a_i x}.$$

Assume that each agent i is risk averse, that is, $a_i > 0$. Also assume that $a_1 \leq a_2$.

Each agent i **invests** $s_i \in \mathbb{R}_+$ units of wealth on a risky company. The company has value $V_0 = \sum_N s_i$ after investments. With probability $p \in [0, 1]$, this value brings a return $r \in [0, 1]$ and becomes $V_1^h = (1 + r)V_0$. With the remaining probability $(1 - p)$, the company goes bankrupt and its value becomes $V_1^l = \beta V_0$ for $\beta \in (0, 1)$.

In case of bankruptcy, the value of the firm is allocated among the agents according to a prespecified bankruptcy rule. Formally, a **bankruptcy problem** is a vector of claims $c = (c_1, c_2) \in \mathbb{R}^2_+$ and an endowment $E \in \mathbb{R}_+$ satisfying $\sum_N c_i \geq E$. Let $\mathcal{B}$ be the class of all bankruptcy problems. A **bankruptcy rule** F assigns each $(c, E) \in \mathcal{B}$ to an allocation $x \in \mathbb{R}^2_+$ satisfying $\sum x_i = E$. In this paper, we will focus on the following four bankruptcy rules.

The **Proportional Rule (PRO)** is defined as follows: for each $i \in N$,

$$F_i^P(c, E) = \lambda c_i$$

where $\lambda \in \mathbb{R}_+$ satisfies $\lambda \sum_N c_i = E$. The **Constrained Equal Awards rule (CEA)** is defined as

$$F_i^A(c, E) = \min\{c_i, \rho\}$$

where $\rho \in \mathbb{R}_+$ satisfies $\sum_N \min\{c_i, \rho\} = E$. The **Constrained Equal Losses rule (CEL)** is defined as

$$F_i^L(c, E) = \max\{c_i - \rho, 0\}$$

where $\rho \in \mathbb{R}_+$ satisfies $\sum_N \max\{c_i - \rho, 0\} = E$. Finally, the **Talmudic rule (TAL)** can be defined as a mixture of the *PRO*, *CEA* and *CEL* rules (*e.g.* see Dagan, Serrano, and Volij (1997)):

$$F_i^T(c, E) = \begin{cases} F_i^A(0.5c, E) & if \quad E < 0.5 \sum_N c_i, \\ F_i^P(c, E) & if \quad E = 0.5 \sum_N c_i. \\ 0.5c_i + F_i^L(0.5c, E - 0.5 \sum_N c_j) & if \quad E > 0.5 \sum_N c_i. \end{cases}$$

In our model, the bankrupt firm retains β fraction of its capital. Thus $E = \beta \sum_N c_i$ is a function of c. As a result we will write $F(c)$ instead of $F(c, E)$.

For each bankruptcy rule F, we analyze the following investment game it induces over the agents. Each $i \in N$ has the strategy set $S_i = \mathbb{R}_+$ from which he chooses an investment level s_i. Let $S = \prod_N S_i$. A strategy profile $s \in S$ corresponds to the following lottery for agent i:

$$\omega_i^F(s) = \begin{cases} (1+r)s_i - s_i & \text{with probability } p, \\ F_i(s) - s_i & \text{with probability } (1-p). \end{cases}$$

Note that $F_i(s) - s_i \leq 0$.

The interpretation is that the agent initially borrows s_i at an interest rate normalized to 1. If the investment is successful, he receives $(1+r)\, s_i$, pays back s_i, and is left with his profit rs_i. In case of bankruptcy, he only receives back $F_i(s)$ and has to pay back s_i, so his net profit becomes $F_i(s) - s_i$. The same lottery is obtained from an environment where each agent i allocates his monetary endowment between a riskless asset (whose return is normalized to 1) and the risky company. In this second interpretation, assume that the agent does not have a liquidity constraint. That is, he is allowed to invest more than his endowment. This assumption only serves to rid us from (the unrealistic) boundary cases where some agents spend all their monetary endowment on the risky firm. Alternatively, one can impose a liquidity constraint but focus on equilibria which are in the interior of the strategy spaces.

Agent i's expected payoff from strategy profile $s \in S$ is thus

$$\begin{aligned} U_i^F(s) &= pu_i(rs_i) + (1-p)u_i(F_i(s) - s_i). \\ &= -pe^{-a_i rs_i} - (1-p)e^{-a_i F_i(s_i, s_{-i}) + a_i s_i} \end{aligned}$$

Let $U^F = \left(U_1^F, U_2^F\right)$. The **investment game induced by the bankruptcy rule F** is then defined as

$$G^F = \langle S, U^F \rangle.$$

Let $\epsilon(G^F)$ denote the set of Nash equilibria of G^F.

3 Equilibria Under Alternative Bankruptcy Rules

We start by analyzing the Nash equilibria of each game.

3.1 Proportional Rule (PRO)

PRO, as a function of the investment levels, is defined as $F_i^P(s) = \beta s_i$. Remember that

$$U_i^F(s) = -pe^{-a_i rs_i} - (1-p)e^{-a_i F_i(s_i, s_{-i}) + a_i s_i}.$$

Thus, agent i's utility under PRO can be written as

$$U_i^P(s) = -pe^{-a_i rs_i} - (1-p)e^{a_i s_i (1-\beta)}.$$

Note that the payoff function of agent i is independent of the other agent's investment level. Thus, so is the best response of agent i. As a result, the investment game under PRO has a dominant strategy equilibrium, as characterized in the following result.

Proposition 1 *The investment game under PRO has a unique dominant strategy equilibrium s^* in which for each $i \in N$, $s_i^* = \max\left\{0, \frac{1}{a_i(r+1-\beta)} \ln\left(\frac{pr}{(1-p)(1-\beta)}\right)\right\}$.*

Note that if $pr > (1-p)(1-\beta)$, both agents choose a positive investment level at the dominant strategy equilibrium. This condition simply compares the return on unit investment in case of success, r, weighted by the probability of succes, p, with the loss incurred on unit investment in case of failure, $(1-\beta)$, weighted by the probability of failure. $(1-p)$. Investing on the firm is optimal if the returns in case of success outweigh the losses incurred in case of failure.

It is interesting to note that the optimal investment level s_i^* is not always increasing in the rate of return in case of success, r. The necessary and sufficient condition for s_i^* to be increasing in r is $s_i^* \leq \frac{1}{a_i r}$. For optimal investment levels higher than $1/a_i r$, agent i decreases his optimal investment level in response to an increase in r.

On the other hand, the optimal investment level is always increasing in the probability of success p and the fraction of the firm that survives bankruptcy β.

Finally note that the optimal investment level s_i^* is decreasing in the agent's degree of risk aversion a_i. Thus, at the dominant strategy equilibrium, the less risk averse agent invests more than the other.

3.2 Constrained Equal Awards Rule (CEA)

CEA, as a function of the investment levels, is defined as $F_i^A(s_1, s_2) = \min\{s_i, \rho\}$ where $\rho \in \mathbb{R}$ satisfies $\sum_N \min\{s_j, \rho\} = \beta \sum_N s_j$. The function F_i^A is written below more explicitly: if agent i invests too little (the first line), he gets full refund and if he invests too much (the third line), the other agent gets full refund. For in-between investment levels (the second line), the liquidation value of the firm is equally allocated between the two agents.

$$F_i^A(s_i, s_j) = \begin{cases} s_i & \text{if } s_i \leq \frac{\beta}{2-\beta} s_j, \\ \frac{\beta}{2}(s_i + s_j) & \text{if } \frac{\beta}{2-\beta} s_j \leq s_i \leq \frac{2-\beta}{\beta} s_j, \\ \beta(s_i + s_j) - s_j & \text{if } s_i \geq \frac{2-\beta}{\beta} s_j. \end{cases}$$

Remember that

$$U_i^F(s) = -pe^{-a_i r s_i} - (1-p)e^{-a_i F_i(s_i, s_{-i}) + a_i s_i}.$$

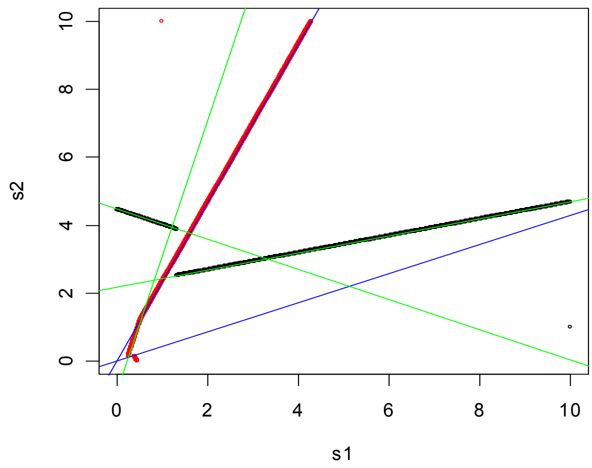


Figure 1: The investment game induced by CEA does not have a Nash equilibrium under the parameter values $p = 0.8$, $\beta = 0.6$, $r = 0.5$, $a_1 = 0.4$, and $a_2 = 4$. The area between the two blue rays satisfies $\frac{\beta}{2-\beta}s_1 \leq s_2 \leq \frac{2-\beta}{\beta}s_1$.

Thus, the utility function of agent i is

$$U_i^A(s_i, s_j) = \begin{cases} -pe^{-a_i r s_i} - (1-p) & \text{if } s_i \leq \frac{\beta}{2-\beta}s_j, \\ -pe^{-a_i r s_i} - (1-p)e^{-a_i \frac{\beta}{2}(s_i+s_j)+a_i s_i} & \text{if } \frac{\beta}{2-\beta}s_j \leq s_i \leq \frac{2-\beta}{\beta}s_j, \\ -pe^{-a_i r s_i} - (1-p)e^{a_i(1-\beta)(s_i+s_j)} & \text{if } s_i \geq \frac{2-\beta}{\beta}s_j. \end{cases}$$

Note that, agent i's payoff function is different in each one of the three intervals. To determine his best response to s_{-i}, agent i compares his payoffs from each one of his optimal choices in these intervals and picks the one(s) that yield the highest payoff. Lemma 10 in the appendix constructs each agent's best response correspondence under the CEA rule. The following example uses it to show that under CEA, the investment game need not have an equilibrium.

Example 1 *Let $N = \{1, 2\}$. Let $p = 0.8$, $\beta = 0.6$, and $r = 0.5$. Let $a_1 = 0.4$ and $a_2 = 4$. Figure 1 shows the agents' best response correspondences under these parameter values. The red best response curve belongs to agent 1 and the black belongs to agent 2.As can be observed in the figure, the two best response curves do not intersect. Thus, the investment game has no Nash equilibria under these parameter values.*

We also observe that there is no Nash equilibrium where an agent, in case of bankruptcy, is fully refunded. This means that the two best response curves can only intersect in the middle section of Figure 1, that is, when $\frac{\beta}{2-\beta}s_j < s_i < \frac{2-\beta}{\beta}s_j$.

Lemma 2 *There is no Nash equilibrium* (s_1^*, s_2^*) *where for some* $i \in N, F_i^A(s_1^*, s_2^*) = s_i^* > 0$.

By Lemma 2, we know that at any equilibrium s^* of the investment game under CEA, the bankruptcy rule CEA awards the agents the following allocation:

$$F^A(s^*) = (\frac{\beta}{2}\sum_N s_j^*, \frac{\beta}{2}\sum_N s_j^*).$$

We make use of this fact in the following proposition.

Proposition 3 *If* s^* *is a Nash equilibrium of the investment game under CEA, then for each* $i \in N$, $s_i^* = \max\left\{0, \frac{\left(1+r-\frac{\beta}{2}\right)a_j+\frac{\beta}{2}a_i}{a_i a_j (r+1)(r-\beta+1)} \ln\left(\frac{pr}{(1-p)\left(1-\frac{\beta}{2}\right)}\right)\right\}$.

Note that if $pr > (1-p)(1-\frac{\beta}{2})$, both agents choose a positive investment level at the Nash equilibrium of the investment game under CEA. This condition simply compares the return on unit investment in case of success, r, weighted by the probability of succes, p, with the loss incurred on unit investment in case of failure, $\left(1-\frac{\beta}{2}\right)$, weighted by the probability of failure, $(1-p)$. Investing on the firm is optimal if the returns in case of success outweigh the losses incurred in case of failure. Note that, the only difference between this condition and the one discussed for PRO at the end of last section is the $\frac{\beta}{2}$ (instead of β for PRO). The reason is simple. Under PRO, one additional unit of investment, in case of bankruptcy brought a return of β. However, since the CEA rule divides the return equally among the two agent, one additional unit of investment under CEA, will only bring a return of $\frac{\beta}{2}$ (the other half being transfered to the other agent).

Similar to the case of PRO, the optimal investment level under CEA, s_i^* is not always increasing in the rate of return in case of success, r. The necessary and sufficient condition for s_i^* to be increasing in r is

$$s_i^* \leq \frac{2}{a_i a_j r}\frac{\left(\left(1+r-\frac{\beta}{2}\right)a_j + \frac{\beta}{2}a_i\right)^2}{\left((1+r)^2+(1+r-\beta)^2\right)a_j + (2r+2-\beta)\beta a_i}.$$

For optimal investment levels higher than this critical value, agent i decreases his optimal investment level in response to an increase in r.

On the other hand, the optimal investment level is always increasing in the probability of success p and the fraction of the firm that survives bankruptcy β. This is also similar to PRO.

Note that the optimal investment level s_i^* is decreasing in the agent's degree of risk aversion a_i. Thus, at the Nash equilibrium, the less risk averse agent invests more than the other.

Finally note that agent i's optimal investment level s_i^* also depends on the other agent's degree of risk aversion a_j. (This relationship did not exist under the Proportional rule.) As a_j increases, that is, as the other agent gets more risk averse, agent i's optimal investment level decreases. The simple reason is as follows. As a_j increases' agent j decreases his investment level and thus, the amount that will be shared in case of bankruptcy. However, since agent j continues to get an equal share as agent i, agent i's return to investment in case of bankruptcy decreases in response. Thus, agent i's equilibrium investment level also decreases.

3.3 Constrained Equal Losses Rule (CEL)

CEL, as a function of the investment levels, is defined as $F_i^L(s) = \max\{s_i - \rho, 0\}$ where $\rho \in \mathbb{R}_+$ satisfies $\sum \max\{s_j - \rho, 0\} = \beta \sum s_j$. The function F_i^L is written below more explicitly: if agent i invests too little (first line), he gets zero refund and if he invests too much (third line), the other agent gets zero refund. For in-between investment levels (the second line), the liquidation value of the firm is allocated to equate the agents' losses.

$$F_i^L(s_i, s_j) = \begin{cases} 0 & \text{if } s_i \leq \frac{1-\beta}{1+\beta} s_j, \\ s_i - \frac{1-\beta}{2}(s_i + s_j) & \text{if } \frac{1-\beta}{1+\beta} s_j \leq s_i \leq \frac{1+\beta}{1-\beta} s_j, \\ \beta(s_i + s_j) & \text{if } s_i \geq \frac{1+\beta}{1-\beta} s_j. \end{cases}$$

Remember that

$$U_i^F(s) = -pe^{-a_i r s_i} - (1-p)e^{-a_i F_i(s_i, s_{-i}) + a_i s_i}.$$

Thus, the utility function of agent i becomes

$$U_i^L(s_i, s_j) = \begin{cases} -pe^{-a_i r s_i} - (1-p)e^{a_i s_i} & \text{if } s_i \leq \frac{1-\beta}{1+\beta} s_j \\ -pe^{-a_i r s_i} - (1-p)e^{a_i \frac{1-\beta}{2}(s_i + s_j)} & \text{if } \frac{1-\beta}{1+\beta} s_j \leq s_i \leq \frac{1+\beta}{1-\beta} s_j \\ -pe^{-a_i r s_i} - (1-p)e^{a_i(1-\beta)s_i - a_i \beta s_j} & \text{if } s_i \geq \frac{1+\beta}{1-\beta} s_j \end{cases}$$

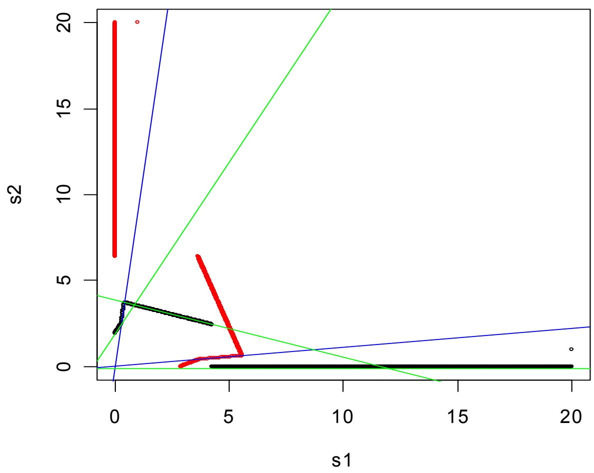


Figure 2: The investment game induced by CEL does not have a Nash equilibrium under the parameter values $p = 0.8$, $\beta = 0.8$, $r = 0.2$, $a_1 = 1.2$ and $a_2 = 1.8$. The area between the two blue rays satisfies $\frac{1-\beta}{1+\beta}s_1 \leq s_2 \leq \frac{1+\beta}{1-\beta}s_1$.

Note that, agent i's payoff function is different in each one of the three intervals. To determine his best response to s_{-i}, agent i compares his payoffs from each one of his optimal choices in these intervals and picks the one(s) that yield the highest payoff. The following example uses lemmas 11 and 12 to show that under CEL, the investment game need not have an equilibrium.

Example 2 *Let $N = \{1, 2\}$. Let $p = 0.8$, $\beta = 0.8$, and $r = 0.2$. Let $a_1 = 1.2$ and $a_2 = 1.8$. The figure below shows the agents' best response correspondences under these parameter values. The red best response curve belongs to agent 1 and the black belongs to agent 2. As can be observed in the figure, the two best response curves do not intersect. Thus, the investment game has no Nash equilibria under these parameter values.*

The following proposition determines the form of Nash equilibria.

Proposition 4 *If s^* is a Nash equilibrium of the investment game under CEL, then it has either one of the following forms.*

Type 1. $s_i^* = \max\left\{0, \frac{(2r-\beta+1)}{2r(r-\beta+1)}\left(\frac{1}{a_i} - \frac{(1-\beta)}{(1-\beta+2r)}\frac{1}{a_j}\right)\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)\right\}$ *for both* $i \in N$ *or*

Type 2. $s_i^* = 0$ *and* $s_j^* = \frac{1}{a_j(1+r-\beta)}\ln\left(\frac{pr}{(1-p)(1-\beta)}\right)$, *or*

Type 3. $s_i^* = \frac{1}{a_i(1+r)}\ln\left(\frac{pr}{1-p}\right)$ *and* $s_j^* = \frac{1}{a_j(r-\beta+1)}\ln\left(\frac{pr}{(1-\beta)(1-p)}\right) + \frac{\beta}{a_i(r+1)(r-\beta+1)}\ln\left(\frac{pr}{1-p}\right)$.

Let us analyze how these three equilibria change in response to a change in the parameters.

3.3.1 Type 1 Equilibrium

For an equilibrium of this kind to exist, the risk aversion levels of the two agents must be sufficiently close. More precisely,

$$1 \leq \frac{a_2}{a_1} < \frac{(1-\beta+2r)}{(1-\beta)}.$$

Note that, then $\frac{1}{a_i} - \frac{(1-\beta)}{(1-\beta+2r)}\frac{1}{a_j} > 0$ for both agents. Therefore, if $pr > (1-p)(\frac{1-\beta}{2})$, both agents choose a positive investment level at the Nash equilibrium of the investment game under CEL. This condition simply compares the return on unit investment in case of success, r, weighted by the probability of succes, p, with the loss incurred on unit investment in case of failure, $\left(\frac{1-\beta}{2}\right)$, weighted by the probability of failure, $(1-p)$. Investing on the firm is optimal if the returns in case of success outweigh the losses incurred in case of failure. Note that, the only difference between this condition and the one discussed for PRO at the end of last section is the $\left(\frac{1-\beta}{2}\right)$ term (which was $\left(1-\frac{\beta}{2}\right)$ in case of CEA and $(1-\beta)$ in case of PRO, as discussed in the previous sections). The reason is simple. The CEL rule, in a Type 1 equilibrium, divides the total loss, $(1-\beta)$, equally among the two agents.

Similar to the case of PRO, the optimal investment level under CEA, s_i^* is not always increasing in the rate of return in case of success, r. The necessary and sufficient condition for s_i^* to be increasing in r is

$$s_i^* > \frac{1}{2ra_ia_j(r-\beta+1)^2}\left(a_jr^2 + a_j(r+1-\beta)^2 - a_i(1-\beta)(1-\beta+2r)\right)\left(\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)\right)^2.$$

For optimal investment levels lower than this critical value, agent i decreases his optimal investment level in response to an increase in r. Note that, due to the second term, this critical value can be negative. In that case, agent i always increases his equilibrium investment in response to an increase in the interest rate r.

On the other hand, the optimal investment level is always increasing in the probability of success p and the fraction of the firm that survives bankruptcy β. This is also similar to PRO.

Note that the optimal investment level s_i^* is decreasing in the agent's degree of risk aversion a_i. Thus, at the Nash equilibrium, the less risk averse agent invests more than the other.

Finally note that agent i's optimal investment level s_i^* also depends on the other agent's degree of risk aversion a_j. (This relationship did not exist under the Proportional rule.) As a_j increases, that is, as the other agent gets more risk averse, agent i's optimal investment level decreases. The simple reason is as follows. As a_j increases' agent j decreases his investment level and thus, the amount that will be shared in case of bankruptcy. However, since agent j continues to get an equal share as agent i, agent i's return to investment in case of bankruptcy decreases inresponse. Thus, agent i's equilibrium investment level also decreases.

4 Comparison of Equilibria

Note that, in equilibrium under PRO and CEA rules, a typical agent chooses the following investment levels:

$$\begin{aligned} s_i^P &= \max\left\{0, \frac{1}{a_i\left(r+1-\beta\right)}\ln\left(\frac{pr}{(1-p)(1-\beta)}\right)\right\}, \\ s_i^A &= \max\left\{0, \frac{(r+1)-\frac{\beta}{2}+\frac{\beta}{2}a_i\frac{1}{a_j}}{a_i\left(r+1\right)\left(r+1-\beta\right)}\ln\left(\frac{pr}{\left(1-p\right)\left(1-\frac{\beta}{2}\right)}\right)\right\} \end{aligned}$$

Under the CEL rule, the equilibrium investment level depends on the type of equilibria and agent. The investment level

$$s_i^{L,1} = \max\left\{0, \frac{(2r-\beta+1)}{2r\left(r-\beta+1\right)}\left(\frac{1}{a_i}-\frac{(1-\beta)}{(1-\beta+2r)}\frac{1}{a_j}\right)\ln\left(\frac{2pr}{\left(1-p\right)\left(1-\beta\right)}\right)\right\}$$

is observed at a "type 1" equilibrium, as itemized in Proposition ***. The investment level

$$s_i^{L,2} = \frac{1}{a_i\left(1+r-\beta\right)}\ln\left(\frac{pr}{\left(1-p\right)\left(1-\beta\right)}\right)$$

is obtained at an equilibrium when the other agent chooses zero investment. Finally, the investment levels

$$\begin{aligned} s_i^{L,3,1} &= \frac{1}{a_i(1+r)}\ln\left(\frac{pr}{1-p}\right) \\ s_j^{L,3,2} &= \frac{1}{a_j(r-\beta+1)}\ln\left(\frac{pr}{(1-\beta)(1-p)}\right)+\frac{\beta}{a_i(r+1)(r-\beta+1)}\ln\left(\frac{pr}{1-p}\right) \end{aligned}$$

are chosen in a type 3 equilibrium.

4.1 Comparison of CEA and Pro

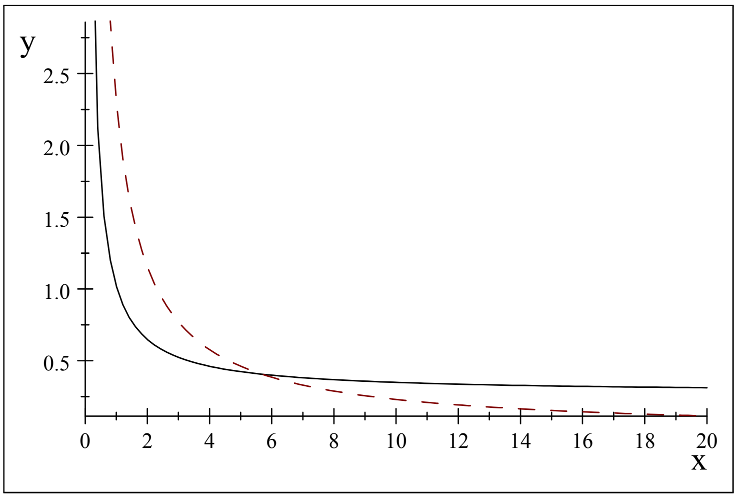
Note that $\frac{pr}{(1-p)\left(1-\frac{\beta}{2}\right)} < \frac{pr}{(1-p)(1-\beta)}$. Thus, if agent i chooses positive investment under the CEA rule, he will also do so under PRO rule. Also note that, $\frac{pr}{(1-p)\left(1-\frac{\beta}{2}\right)}$ and $\frac{pr}{(1-p)(1-\beta)}$ are independent of the agents' (personal) risk aversion parameters. Therefore, under both rules, either all agents choose positive investment or all agents choose zero investment.

Now note that, under both rules, the investment choice of an agent is decreasing in his risk aversion parameter. What is harder to see though is that there is a critical risk aversion level for agent i,

$$a_i^* = \left(\frac{\ln\left(\frac{pr}{(1-p)(1-\beta)}\right)}{\ln\left(\frac{pr}{(1-p)\left(1-\frac{\beta}{2}\right)}\right)}-1+\frac{\beta}{2(r+1)}\right)\frac{2(r+1)}{\beta}a_j$$

for which the equilibrium investment level coincides under the two rule. For smaller risk aversion level, PRO delivers more investment and for higher risk aversion levels, it delivers less investment than the CEA rule, as demonstrated in the following example.

Example 3 *Let $r = 0.3$, $p = 0.8$, $\beta = 0.7$, and $a_j = 1$. Agent i's equilibrium investment as a function of his risk aversion a_i is the solid curve under the CEA rule and the dotted curve under PRO.*



Agent i's equilibrium investment as a function of his risk aversion a_i is the solid curve under the CEA rule and the dotted curve under PRO.

We will next compare the two rules in terms of the total investment they induce in the economy. Since less risk averse agents invest much more than the more risk averse ones (under both rules) and since these agents invest more under PRO rule, we obtain the following result.

Proposition 5 *If* $\ln\left(\frac{pr}{(1-p)(1-\beta)}\right) > 0$, *PRO induces a higher total investment than CEA. If* $\ln\left(\frac{pr}{(1-p)(1-\beta)}\right) \leq 0$, *both rules induce a zero investment level.*

Proof. Note that with PRO,

$$\epsilon(G^P) = \begin{cases} (s_1^P, s_2^P) & if \quad \ln\left(\frac{pr}{(1-p)(1-\beta)}\right) \geq 0, \\ (0,0) & \text{otherwise.} \end{cases}$$

and with the CEA rule,

$$\epsilon(G^A) = \begin{cases} (s_1^A, s_2^A) & if \quad \ln\left(\frac{pr}{(1-p)(1-\frac{\beta}{n})}\right) \geq 0, \\ (0,0) & \text{otherwise.} \end{cases}$$

The parameter space can be partitioned into three parts. We will formulate each partition as a case below.

Case 1. $pr \leq (1-p)(1-\beta)$

Note that then $\ln\left(\frac{pr}{(1-p)(1-\beta)}\right) \leq 0$ and thus, $\epsilon(G^P) = \mathbf{0}$. Also, $\ln\left(\frac{pr}{(1-p)\left(1-\frac{\beta}{n}\right)}\right) < 0$ holds and thus, $\epsilon(G^A) = \mathbf{0}$.

Case 2. $(1-p)(1-\beta) < pr \leq (1-p)(1-\frac{\beta}{2})$

Note that then $\ln\left(\frac{pr}{(1-p)\left(1-\frac{\beta}{2}\right)}\right) \leq 0$ holds and thus, $\epsilon(G^A) = \mathbf{0}$. On the other hand, $\ln\left(\frac{pr}{(1-p)(1-\beta)}\right) > 0$ and thus, $\epsilon(G^P) > \mathbf{0}$.

Case 3. $(1-p)(1-\frac{\beta}{2}) < pr$

In this case, $\ln\left(\frac{pr}{(1-p)(1-\beta)}\right) > \ln\left(\frac{pr}{(1-p)\left(1-\frac{\beta}{2}\right)}\right) > 0$. Therefore, $\epsilon(G^A) > \mathbf{0}$ and $\epsilon(G^P) > \mathbf{0}$.

Now the total investment level induced by PRO is

$$\begin{aligned} I^P &= \sum_N \frac{1}{a_i\,(r+1-\beta)} \ln\left(\frac{pr}{(1-p)(1-\beta)}\right) \\ &= \left(\sum_N \frac{1}{a_i}\right) \frac{1}{(r+1-\beta)} \ln\left(\frac{pr}{(1-p)(1-\beta)}\right). \end{aligned}$$

Similarly, the total investment level induced by CEA is

$$\begin{aligned} I^A &= \sum_N \frac{\left(1+r-\frac{\beta}{2}\right)a_j + \frac{\beta}{2}a_i}{a_i a_j\,(r+1)\,(r-\beta+1)} \ln\left(\frac{pr}{(1-p)\left(1-\frac{\beta}{2}\right)}\right) \\ &= \left(\sum_N \frac{1}{a_i}\right) \frac{1}{(r+1-\beta)} \ln\left(\frac{pr}{(1-p)\left(1-\frac{\beta}{2}\right)}\right). \end{aligned}$$

The difference is then

$$\begin{aligned} I^P - I^A &= \left(\sum_N \frac{1}{a_i}\right) \frac{1}{(r+1-\beta)} \left(\ln\left(\frac{pr}{(1-p)(1-\beta)}\right) - \ln\left(\frac{pr}{(1-p)\left(1-\frac{\beta}{2}\right)}\right)\right) \\ &= \left(\sum_N \frac{1}{a_i}\right) \frac{1}{(r+1-\beta)} \ln\left(\frac{\left(1-\frac{\beta}{n}\right)}{(1-\beta)}\right) > 0. \end{aligned}$$

■

4.2 Comparison of CEL and Pro

Note that $\frac{pr}{(1-p)} < \frac{pr}{(1-p)(1-\beta)} < \frac{2pr}{(1-p)(1-\beta)}$. Thus, if agent i chooses positive investment under PRO, he will also do so under a Type 1 equilibrium under the CEL rule. Note that, agent

$i's$ investment under PRO is equal to his investment in a type 2 equilibrium under the CEL rule. Finally, if there is a Type 3 equilibrium, then $\frac{pr}{(1-p)} > 1$. Thus, agent i also chooses positive investment under PRO. Also note that these observations are independent of the agents' (personal) risk aversion parameters. Therefore, under both rules, either all agents choose positive investment or all agents choose zero investment.

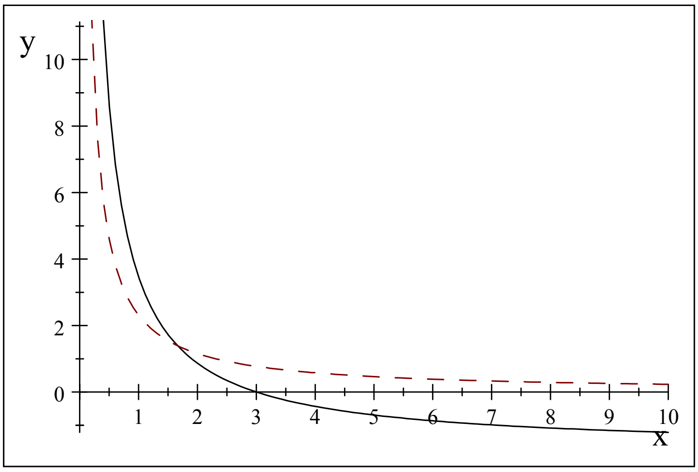
4.2.1 Comparison of Pro with a Type 1 equilibrium under CEL

Assume $\frac{pr}{(1-p)(1-\beta)} > 1$. Then agent i chooses a positive investment level under both rules. Also, under both rules the investment choice of an agent is decreasing in his risk aversion parameter. What is harder to see though is that there is a critical risk aversion level for agent i,

$$a_i^* = \left(\frac{(1-\beta+2r)}{(1-\beta)} - \frac{2r \ln\left(\frac{pr}{(1-p)(1-\beta)}\right)}{(1-\beta)\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)} \right) a_j$$

for which the equilibrium investment level coincides under the two rule. In contrast to the CEA case, for smaller risk aversion level, PRO delivers less investment and for higher risk aversion levels, it delivers more investment than the CEL rule, as demonstrated in the following example.

Example 4 *Let $r = 0.3$, $p = 0.8$, $\beta = 0.7$, and $a_j = 1$. Agent i's equilibrium investment as a function of his risk aversion a_i is the solid curve under the CEL rule and the dotted curve under PRO.*



Agent i's equilibrium investment as a function of his risk aversion a_i is the solid curve under the CEL rule and the dotted curve under PRO.

We will next compare the two rules in terms of the total investment they induce in the economy. Since less risk averse agents invest much more than the more risk averse ones (under both rules) and since these agents invest more under the CEL rule, we obtain the following result.

Proposition 6 *If* $\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right) > 0$, *Type 1 equilibrium under the CEL rule induces a higher total investment than PRO. If* $\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right) \leq 0$, *both rules induce a zero investment level.*

Proof. The parameter space can be partitioned into three parts. We will formulate each partition as a case below.

Case 1. $\mathbf{2}pr \leq (1-p)(1-\beta)$

Note that then $\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right) \leq 0$ and thus, $\epsilon(G^P) = \mathbf{0}$ and $\epsilon(G^L) = \mathbf{0}$.

Case 2. $(1-p)(1-\beta) < 2pr \leq 2(1-p)(1-\beta)$

Note that then $\ln\left(\frac{pr}{(1-p)(1-\beta)}\right) \leq 0$ holds and thus, $\epsilon(G^P) = \mathbf{0}$. On the other hand, $\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right) > 0$ and thus, $\epsilon(G^L) > \mathbf{0}$.

Case 3. $(1-p)(1-\beta) < pr$

In this case, $\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right) > \ln\left(\frac{pr}{(1-p)(1-\beta)}\right) > 0$. Therefore, $\epsilon(G^L) > \mathbf{0}$ and $\epsilon(G^P) > \mathbf{0}$.
Now the total investment level induced by PRO is

$$\begin{aligned} I^P &= \sum_N \frac{1}{a_i\,(r+1-\beta)} \ln\left(\frac{pr}{(1-p)(1-\beta)}\right) \\ &= \left(\sum_N \frac{1}{a_i}\right) \frac{1}{(r+1-\beta)} \ln\left(\frac{pr}{(1-p)(1-\beta)}\right). \end{aligned}$$

Similarly, the total investment level induced by the CEL rule is

$$\begin{aligned} I^L &= \sum_N \frac{(2r-\beta+1)}{2r\,(r-\beta+1)} \left(\frac{1}{a_i} - \frac{(1-\beta)}{(1-\beta+2r)}\frac{1}{a_j}\right) \ln\left(\frac{2pr}{(1-p)\,(1-\beta)}\right) \\ &= \left(\sum_N \frac{1}{a_i}\right) \frac{1}{(r+1-\beta)} \ln\left(\frac{2pr}{(1-p)\,(1-\beta)}\right). \end{aligned}$$

The difference is then

$$
\begin{aligned}
I^P - I^L &= \left(\sum_N \frac{1}{a_i}\right) \frac{1}{(r+1-\beta)} \left(\ln\left(\frac{pr}{(1-p)(1-\beta)}\right) - \ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)\right) \\
&= \left(\sum_N \frac{1}{a_i}\right) \frac{1}{(r+1-\beta)} \ln\left(\frac{1}{2}\right) < 0.
\end{aligned}
$$

■

4.2.2 Comparison of Pro with a Type 2 equilibrium under CEL

In a Type 2 equilibrium under the CEL rule, agent i's investment level is given by

$$
s_i^{L,2} = \frac{1}{a_i\,(1+r-\beta)} \ln\left(\frac{pr}{(1-p)\,(1-\beta)}\right).
$$

This expression is identical to the one obtained under PRO. However, under CEL, the other agent chooses a zero investment level while under PRO both agents choose this investment level. Therefore, we have the following proposition.

Proposition 7 *If* $\ln\left(\frac{pr}{(1-p)(1-\beta)}\right) > 0$, *the equilibrium under PRO induces a higher total investment than a Type 2 equilibrium of the CEL rule. If* $\ln\left(\frac{pr}{(1-p)(1-\beta)}\right) \leq 0$, *both rules induce a zero investment level.*

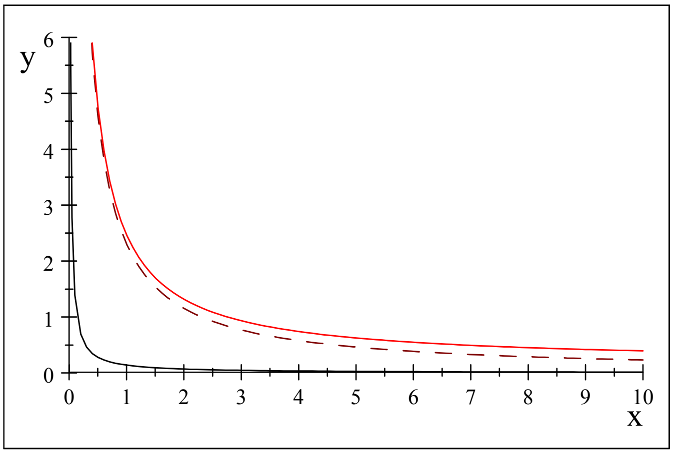
4.2.3 Comparison of Pro with a Type 3 equilibrium under CEL

Assume $\frac{pr}{(1-p)} > 1$. Then all agents choose a positive investment level under both rules. Also, under both rules the investment choice of an agent is decreasing in his risk aversion parameter.

First observe that $s_i^P > s_i^{L,3,1}$. That is, the small investor under the CEL type 3 equilibrium invests much less than in PRO. On the other hand, $s_i^P < s_i^{L,3,2}$. That is, the big investor under the CEL type 3 equilibrium invests much more than in PRO.

The following example demonstrates the three investment levels as a function of risk aversion.

Example 5 *Let* $r = 0.3$, $p = 0.8$, $\beta = 0.7$, *and* $a_j = 1$. *Agent* i*'s equilibrium investment as a function of his risk aversion* a_i *is the two solid curves under the CEL rule: the black curve* *is the small investor and the red one, the big investor. The dotted curve is the investment level under PRO.*



Agent i's equilibrium investment as a function of his risk aversion a_i is the two solid curves under the CEL rule: the black curve is the small investor and the red one, the big investor. The dotted curve is the investment level under PRO.

We will next compare the two rules in terms of the total investment they induce in the economy.

Proposition 8 *Assume* $\ln\left(\frac{pr}{(1-p)}\right) > 0$. *Then, the type 3 equilibrium under the CEL rule induces a higher total investment than an equilibrium under PRO.*

Proof. The total investment level induced by PRO is

$$\begin{aligned} I^P &= \sum_N \frac{1}{a_i\,(r+1-\beta)} \ln\left(\frac{pr}{(1-p)(1-\beta)}\right) \\ &= \left(\sum_N \frac{1}{a_i}\right) \frac{1}{(r+1-\beta)} \ln\left(\frac{pr}{(1-p)(1-\beta)}\right). \end{aligned}$$

Taking agent i to be the small investor, the total investment level induced by the CEL rule is

$$\begin{aligned} I^L &= \frac{1}{a_i\,(1+r)} \ln\left(\frac{pr}{1-p}\right) + \frac{1}{a_j\,(r-\beta+1)} \ln\left(\frac{pr}{(1-\beta)\,(1-p)}\right) + \frac{\beta}{a_i\,(r+1)\,(r-\beta+1)} \ln\left(\frac{pr}{1-p}\right) \\ &= \left(\sum_N \frac{1}{a_i}\right) \frac{1}{(r+1-\beta)} \ln\left(\frac{pr}{(1-p)(1-\beta)}\right) + \frac{1}{a_i\,(r-\beta+1)} \ln(1-\beta) \end{aligned}$$

The difference is then

$$I^P - I^L = \frac{1}{a_i\left(r - \beta + 1\right)} \ln\left(1 - \beta\right) < 0.$$

■

4.3 Comparison of CEA and CEL

The following proposition compares total investment under the CEL and CEA rules.

Proposition 9 *Type 1 and Type 3 equilibria under the CEL rule induce more total investment than CEA. Type 2 equilibrium under CEL induces more total investment than CEA if and only if*

$$\frac{a_j}{a_i} > \frac{\ln\left(\frac{pr}{(1-p)\left(1-\frac{\beta}{2}\right)}\right)}{\ln\left(\frac{1-\frac{\beta}{2}}{1-\beta}\right)}.$$

Proof. Under Type 1 and Type 3 equilibria under the CEL rule, Propositions 6 and 8 show that there is more investment than under PRO. Since, by Proposition 5, PRO always induces more total investment than the CEA rule, the Type 1 and Type 3 equilibria of the CEL rule exceed the CEA rule as well.

Total investment in a Type 2 equilibrium under the CEL rule is

$$I^L = \frac{1}{a_i\left(1 + r - \beta\right)} \ln\left(\frac{pr}{\left(1-p\right)\left(1-\beta\right)}\right).$$

Total investment under the CEA rule is

$$I^A = \left(\frac{1}{a_i} + \frac{1}{a_j}\right) \frac{1}{\left(r + 1 - \beta\right)} \ln\left(\frac{pr}{\left(1-p\right)\left(1-\frac{\beta}{2}\right)}\right).$$

Now

$$I^L - I^A = \frac{1}{a_i\left(1 + r - \beta\right)} \ln\left(\frac{1-\frac{\beta}{2}}{1-\beta}\right) - \frac{1}{a_j\left(r + 1 - \beta\right)} \ln\left(\frac{pr}{\left(1-p\right)\left(1-\frac{\beta}{2}\right)}\right).$$

The sign of this expression is not clear. However, note that $I^L - I^A$ is increasing in a_j (where agent j is the one who chooses zero investment under the Type 2 Equilibrium of the CEL rule). Also, $\lim_{a_j \to 0} I^L - I^A < 0$. Solving for $I^L - I^A = 0$ we obtain the cutoff. If agent j is sufficiently more risk averse than agent i, that is if

$$\frac{a_j}{a_i} > \frac{\ln\left(\frac{pr}{(1-p)\left(1-\frac{\beta}{2}\right)}\right)}{\ln\left(\frac{1-\frac{\beta}{2}}{1-\beta}\right)},$$

then $I^L - I^A > 0$, the CEL rule induces more investment than the CEA rule. ■

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5 Appendix

This section contains the proofs.

5.1 Proportional Rule

Proof. (Proposition 1) We first construct the best response correspondence of agent i. The first derivative of agent i's utility is

$$\frac{\partial U_i^P(s)}{\partial s_i} = -a_i\left(1-p\right)\left(1-\beta\right)e^{a_i s_i (1-\beta)} + a_i p r e^{-a_i r s_i}$$

and the second derivative is

$$\frac{\partial^2 U_i^P(s)}{\partial s_i^2} = -a_i^2\left(1-\beta\right)^2\left(1-p\right)e^{a_i s_i (1-\beta)} - a_i^2 p r^2 e^{-a_i r s_i} < 0.$$

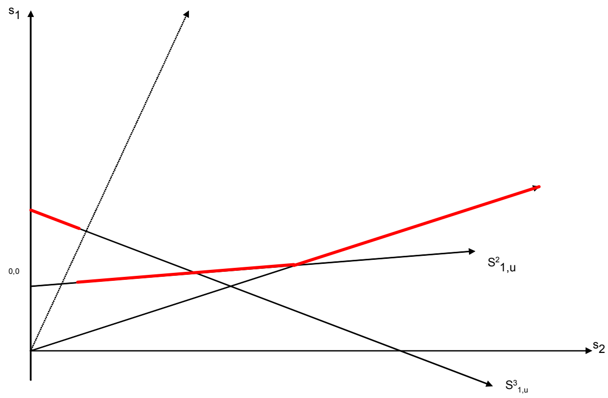

Figure 3: A typical best response correspondence under CEA.

Equating the first derivative to zero, we obtain

$$\overline{s}_i = \frac{1}{a_i\,(r+1-\beta)} \ln\left(\frac{pr}{(1-p)(1-\beta)}\right).$$

The best response function of agent i can then be written

$$BR_i(s_{-i}) = \begin{cases} \overline{s}_i \quad if \quad \ln\left(\frac{pr}{(1-p)(1-\beta)}\right) \geq 0, \\ 0 \qquad\qquad \text{otherwise.} \end{cases}$$

Note that this expression is independent of s_j. So it in fact defines a strictly dominant strategy for each agent i. Thus, the investment game under PRO has a unique dominant strategy equilibrium s^* in which for each $i \in N$, $s_i^* = \max\{0, \overline{s}_i\}$. ■

5.2 CEA Rule

The following lemma characterizes agent i's best response function under the CEA rule. See Figure 3 for a typical configuration.

Lemma 10 *For each set of parameter values, there are* $0 \leq K_1 \leq K_2$ *such that*

$$BR_i(s_j) = \begin{cases} \frac{\beta}{2-\beta} s_j & if \quad \frac{pr}{(1-p)(1-\beta)} < 1, \\ \frac{\ln[pr/(1-p)(1-\beta)]}{a_i(1+r-\beta)} - \frac{1-\beta}{(1+r-\beta)} s_j & if \quad \frac{pr}{(1-p)(1-\beta)} \geq 1 \text{ and } 0 \leq s_j \leq K_1, \\ \frac{\ln[pr/(1-p)(1-\beta/2)]}{a_i(1+r-\beta/2)} + \frac{\beta}{2(1+r-\beta/2)} s_j & if \quad \frac{pr}{(1-p)(1-\beta)} \geq 1 \text{ and } K_1 \leq s_j \leq K_2, \\ \frac{\beta}{2-\beta} s_j & if \quad \frac{pr}{(1-p)(1-\beta)} \geq 1 \text{ and } K_2 \leq s_j. \end{cases}$$

Also, $BR_i(s_j) \neq \frac{2-\beta}{\beta} s_j$ *for all* $s_j \in \mathbb{R}^2_+$.

Proof. We first calculate the maximizer of each expression in the three piece function $U_i^A(s_i, s_j)$.

First assume $s_i \leq \frac{\beta}{2-\beta} s_j$. Then $U_i^A(s_i, s_j) = -pe^{-a_i r s_i} - (1-p)$. The first derivative is

$$\frac{\partial U_i^A(s_i, s_{-i})}{\partial s_i} = \frac{\partial\left(-pe^{-a_i r s_i} - (1-p)\right)}{\partial s_i} = a_i pre^{-a_i r s_i} > 0.$$

Note that agent i's payoff is increasing in s_i. Therefore, the maximizer s_i^1 of U_i^A in the interval $[0, \frac{\beta}{2-\beta} s_j]$ is

$$s_i^1(s_j) = \frac{\beta}{2-\beta} s_j.$$

Now assume $\frac{\beta}{2-\beta} s_j \leq s_i \leq \frac{2-\beta}{\beta} s_j$. Then $U_i^A(s_i, s_j) = -pe^{-a_i r s_i} - (1-p)e^{-a_i \frac{\beta}{2}(s_i + s_j) + a_i s_i}$. The first derivative is

$$\frac{\partial U_i^A(s_i, s_{-i})}{\partial s_i} = -a_i \left(1 - \frac{\beta}{2}\right)(1-p)\, e^{-a_i\left(\frac{\beta}{2} s_j - \left(1 - \frac{\beta}{2}\right) s_i\right)} + pra_i e^{-r a_i s_i}$$

and the second derivative is

$$\frac{\partial^2 U_i^A(s_i, s_{-i})}{\partial s_i^2} = -a_i^2 (1-p) \left(1 - \frac{\beta}{2}\right)^2 e^{-a_i\left(\frac{\beta}{2} s_j - (1 - \frac{\beta}{2}) s_i\right)} - a_i^2 pr^2 e^{-a_i(r s_i + 1)} < 0.$$

Equating the first derivative to zero, we obtain

$$s_i = \frac{\ln\left(\frac{pr}{(1-p)\left(1-\frac{\beta}{2}\right)}\right)}{a_i(1 + r - \frac{\beta}{2})} + \frac{\beta}{2(1 + r - \frac{\beta}{2})} s_j.$$

Therefore, the maximizer s_i^2 of U_i^A in the interval $[\frac{\beta}{2-\beta} s_j, \frac{2-\beta}{\beta} s_j]$ is

$$s_i^2(s_j) = median \left\{ \frac{\beta}{2-\beta} s_j, \frac{2-\beta}{\beta} s_j, \frac{\ln\left(\frac{pr}{(1-p)\left(1-\frac{\beta}{2}\right)}\right)}{a_i(1 + r - \frac{\beta}{2})} + \frac{\beta}{2(1 + r - \frac{\beta}{2})} s_j \right\}.$$

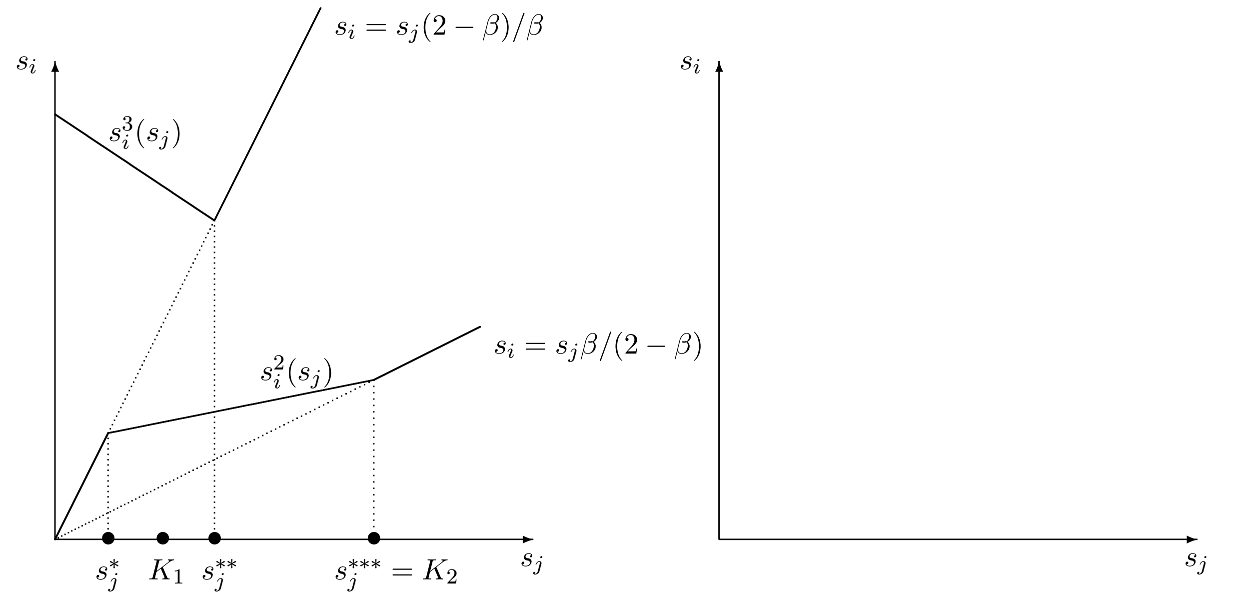

Figure 4: The construction in the proof of Lemma 3.

For the third case, assume $s_i \geq \frac{2-\beta}{\beta}s_j$. Then $U_i^A(s_i, s_j) = -pe^{-a_i r s_i} - (1-p)e^{a_i(1-\beta)(s_i+s_j)}$ and an analysis of the first and the second derivatives gives the unconstrained maximizer

$$s_i = \frac{\ln\left(\frac{pr}{(1-p)(1-\beta)}\right)}{a_i(1+r-\beta)} - \frac{1-\beta}{(1+r-\beta)}s_j.$$

Therefore, the maximizer s_i^3 of U_i^A in the interval $[\frac{2-\beta}{\beta}s_j, \infty)$ is

$$s_i^3(s_j) = \max\left\{\frac{2-\beta}{\beta}s_j, \frac{\ln\left(\frac{pr}{(1-p)(1-\beta)}\right)}{a_i(1+r-\beta)} - \frac{1-\beta}{(1+r-\beta)}s_j\right\}.$$

Comparison of the three local maxima reveals the agent's best response.

Case 1: $\frac{pr}{(1-p)(1-\beta)} < 1$. Then $\ln\left(\frac{pr}{(1-p)(1-\beta)}\right) < 0$ and thus, $s_i^3(s_j) = \frac{2-\beta}{\beta}s_j$. Similarly $\ln\left(\frac{pr}{(1-p)\left(1-\frac{\beta}{2}\right)}\right) < 0$ implies $\frac{\ln\left(\frac{pr}{(1-p)\left(1-\frac{\beta}{2}\right)}\right)}{a_i(1+r-\frac{\beta}{2})} + \frac{\beta}{2(1+r-\frac{\beta}{2})}s_j < \frac{\beta}{2-\beta}s_j$ and thus, $s_i^2(s_j) = \frac{\beta}{2-\beta}s_j$. Since $U^A\left(\frac{\beta}{2-\beta}s_j\right) > U^A(x)$ for all $x \in (\frac{\beta}{2-\beta}s_j, \frac{2-\beta}{\beta}s_j]$, we have $U^A\left(\frac{\beta}{2-\beta}s_j\right) > U^A\left(\frac{2-\beta}{\beta}s_j\right)$. This implies

$$BR_i(s_j) = \frac{\beta}{2-\beta}s_j.$$

Case 2: $\frac{pr}{(1-p)\left(1-\frac{\beta}{2}\right)} \geq 1$. Then $\frac{pr}{(1-p)(1-\beta)} > 1$. Let s_j^* solve $s_i^2(s_j) = \frac{2-\beta}{\beta}s_j$. (This equality has a solution since the slope of s_i^2 is $\frac{\beta}{2(1+r-\frac{\beta}{2})} < \frac{\beta}{2-\beta} < \frac{2-\beta}{\beta}$.) Then $s_j^* = \frac{\beta\ln[pr/(1-p)(1-\beta/2)]}{a_i(2+2r-2\beta-\beta r)}$.

Let s_j^{**} solve $s_i^3(s_j) = \frac{2-\beta}{\beta}s_j$. (This equality has a solution since s_i^3 is downward sloping.) Then $s_j^{**} = \frac{\beta \ln[pr/(1-p)(1-\beta)]}{a_i(2+2r-2\beta-\beta r)}$. Note that $s_j^* < s_j^{**}$.

Now note that $U_i^A\left(s_i^2\left(s_j^*\right)\right) < U_i^A\left(s_i^3\left(s_j^*\right)\right)$ and $U_i^A\left(s_i^2\left(s_j^{**}\right)\right) > U_i^A\left(s_i^3\left(s_j^{**}\right)\right)$. Also, $U_i^A\left(s_i^2(.)\right)$ is increasing and $U_i^A\left(s_i^3(.)\right)$ is decreasing. Therefore, there is a unique $K_1 \in \left(s_j^*, s_j^{**}\right)$ such that $U_i^A\left(s_i^2(K_1)\right) = U_i^A\left(s_i^3(K_1)\right)$. This implies that

$$BR_i(s_j) = \begin{cases} s_i^3(s_j) & if \quad 0 \leq s_j < K_1, \\ \{s_i^2(s_j), s_i^3(s_j)\} & if \quad s_j = K_1, \\ s_i^2(s_j) & if \quad K_1 < s_j. \end{cases}$$

Now let s_j^{***} solve $s_i^2\left(s_j^{***}\right) = \frac{\beta}{2-\beta}s_j^{***}$. (This equality has a solution since the slope of s_i^2 is $\frac{\beta}{2(1+r-\frac{\beta}{2})} < \frac{\beta}{2-\beta}$.) Note that

$$s_i^2(s_j) = \begin{cases} \frac{2-\beta}{\beta}s_j & if \quad 0 \leq s_j < s_j^*, \\ \frac{\ln\left(\frac{pr}{(1-p)\left(1-\frac{\beta}{2}\right)}\right)}{a_i(1+r-\frac{\beta}{2})} + \frac{\beta}{2(1+r-\frac{\beta}{2})}s_j & if \quad s_j^* \leq s_j < s_j^{***}, \\ \frac{\beta}{2-\beta}s_j & if \quad s_j^{***} < s_j. \end{cases}$$

Now, letting $K_2 = \max\left\{K_1, s_j^{***}\right\}$ delivers the desired form for BR_i. Finally, $BR_i(s_j) \neq \frac{2-\beta}{\beta}s_j$ for all $s_j \in \mathbb{R}_+^2$ since $s_j^* < K_1 < s_j^{**}$.

Case 3: $\frac{pr}{(1-p)\left(1-\frac{\beta}{2}\right)} < 1 \leq \frac{pr}{(1-p)(1-\beta)}$. Let $s_j^* = 0$. Let s_j^{**} solve $s_i^3(s_j) = \frac{2-\beta}{\beta}s_j$. (This equality has a solution since s_i^3 is downward sloping.) Then $s_j^{**} = \frac{\beta \ln[pr/(1-p)(1-\beta)]}{a_i(2+2r-2\beta-\beta r)}$. Note that $s_j^* < s_j^{**}$. Then the same arguments in Case 2 deliver the desired form as well as $BR_i(s_j) \neq \frac{2-\beta}{\beta}s_j$ for all $s_j \in \mathbb{R}_+^2$. ■

Proof. (Lemma 2) Suppose there is an equilibrium (s_1^*, s_2^*) and an agent $i \in N$ such that $F_i^A(s_1^*, s_2^*) = s_i^* > 0$. By definition of F^A, $s_i^* \leq \frac{\beta}{2-\beta}s_j^*$. However, by Lemma 10 $s_i^* < \frac{\beta}{2-\beta}s_j^*$ is never a best response. So $s_i^* = \frac{\beta}{2-\beta}s_j^*$. But then, $s_j^* = \frac{2-\beta}{\beta}s_i^*$. This, however, contradicts the second part of Lemma 10. ■

Proof. (Proposition 3) By Lemma 10, for each $i \in N$

$$s_i = \frac{\ln\left(\frac{pr}{(1-p)\left(1-\frac{\beta}{2}\right)}\right)}{a_i(1+r-\frac{\beta}{2})} + \frac{\beta}{2(1+r-\frac{\beta}{2})}s_j.$$

To calculate the Nash equilibrium, we let

$$\sigma_i = \frac{\ln\left(\frac{pr}{(1-p)\left(1-\frac{\beta}{2}\right)}\right)}{a_i(1+r-\frac{\beta}{2})}, \; \rho = \frac{\beta}{2(1+r-\frac{\beta}{2})}$$

and solve the general system:

$$\begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix}.$$

Since $\rho < 1$, there is a unique solution to this system. Solving it gives,

$$s_i^* = \frac{\left(1+r-\frac{\beta}{2}\right)a_j + \frac{\beta}{2}a_i}{a_i a_j \left(r+1\right)\left(r-\beta+1\right)} \ln\left(\frac{pr}{\left(1-p\right)\left(1-\frac{\beta}{2}\right)}\right).$$

Note that, the sign of s_i^* depends on the sign of $\ln\left(\frac{pr}{(1-p)\left(1-\frac{\beta}{n}\right)}\right)$. If this is negative, $s_i^* < 0$ is not a best response. In this case, both agents choose zero investment. Alternatively, if $\ln\left(\frac{pr}{(1-p)\left(1-\frac{\beta}{n}\right)}\right) > 0$, the equilibrium investment level is given by the above expression. Thus the Nash equilibrium can be characterized as follows:

$$\epsilon(G^A) = \begin{cases} (s_1^*, s_2^*) & if \;\; \ln\left(\frac{pr}{(1-p)(1-\frac{\beta}{n})}\right) \geq 0, \\ (0,0) & \text{otherwise.} \end{cases}$$

■

5.3 CEL Rule

The following two lemmas derive agent i's best response correspondence under the CEL rule. The first, Lemma 11 constructs the three possible candidates for optimum.

Lemma 11 *The maximizer of $U_i^L\left(s_i, s_j\right)$ for $s_i \leq \frac{1-\beta}{1+\beta}s_j$ is*

$$s_i^1\left(s_j\right) = median\left\{0, \frac{1-\beta}{1+\beta}s_j, s_{i,u}^1\left(s_j\right)\right\}$$

where

$$s_{i,u}^1\left(s_j\right) = \frac{1}{\left(1+r\right)a_i} \ln\left(\frac{pr}{1-p}\right)$$

is the unconstrained maximizer. The maximizer of $U_i^L(s_i, s_j)$ *for* $\frac{1-\beta}{1+\beta}s_j \leq s_i \leq \frac{1+\beta}{1-\beta}s_j$ *is*

$$s_i^2(s_j) = median\left\{\frac{1-\beta}{1+\beta}s_j, \frac{1+\beta}{1-\beta}s_j, s_{i,u}^2(s_j)\right\}$$

where

$$s_{i,u}^2(s_j) = \frac{2}{a_i(1-\beta+2r)}\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right) - \frac{(1-\beta)}{(1-\beta+2r)}s_j$$

is the unconstrained maximizer. The maximizer of $U_i^L(s_i, s_j)$ *for* $s_i \geq \frac{1+\beta}{1-\beta}s_j$ *is*

$$s_i^3(s_j) = \max\left\{\frac{1+\beta}{1-\beta}s_j, s_{i,u}^3(s_j)\right\}.$$

where

$$s_{i,u}^3(s_j) = \frac{1}{a_i(1+r-\beta)}\ln\left(\frac{pr}{(1-p)(1-\beta)}\right) + \frac{\beta}{(1+r-\beta)}s_j$$

is the unconstrained maximizer.

Proof. First assume $s_i \leq \frac{1-\beta}{1+\beta}s_j$. Then $U_i^L(s_i, s_j) = -pe^{-a_i r s_i} - (1-p)e^{a_i s_i}$. The first derivative is

$$\frac{\partial U_i^L(s_i, s_{-i})}{\partial s_i} = pa_i e^{a_i s_i} - a_i e^{a_i s_i} + pra_i e^{-ra_i s_i}.$$

The second derivative is

$$\frac{\partial^2 U_i^L(s_i, s_{-i})}{\partial s_i^2} = pa_i^2 e^{a_i s_i} - a_i^2 e^{a_i s_i} - pr^2 a_i^2 e^{-ra_i s_i} = -(1-p)a_i^2 e^{a_i s_i} - pr^2 a_i^2 e^{-ra_i s_i} < 0.$$

Equating the first derivative to zero, we obtain the unconstrained maximizer

$$s_{i,u}^1(s_j) = \frac{1}{(1+r)a_i}\ln\left(\frac{pr}{1-p}\right).$$

(Note that this amount is independent of s_j.) Therefore, the maximizer s_i^1 of U_i^L in the interval $[0, \frac{1-\beta}{1+\beta}s_j]$ is $s_i^1(s_j)$.

Now assume $\frac{1-\beta}{1+\beta}s_j \leq s_i \leq \frac{1+\beta}{1-\beta}s_j$. Then $U_i^L(s_i, s_j) = -pe^{-a_i r s_i} - (1-p)e^{a_i \frac{1-\beta}{2}(s_i+s_j)}$. The first derivative is

$$\frac{\partial U_i^L(s_i, s_{-i})}{\partial s_i} = pra_i e^{-ra_i s_i} + \frac{1}{2}(1-p)(\beta-1)a_i e^{-a_i(s_i+s_j)\left(\frac{1}{2}\beta-\frac{1}{2}\right)}$$

and the second derivative is

$$\frac{\partial^2 U_i^L(s_i, s_{-i})}{\partial s_i^2} = -\frac{1}{4}a_i^2\left((\beta-1)^2(1-p)\exp\left(\frac{1}{2}a_i s_i + \frac{1}{2}a_i s_j - \frac{1}{2}\beta a_i s_i - \frac{1}{2}\beta a_i s_j\right) + 4pr^2 e^{-ra_i s_i}\right) < 0.$$

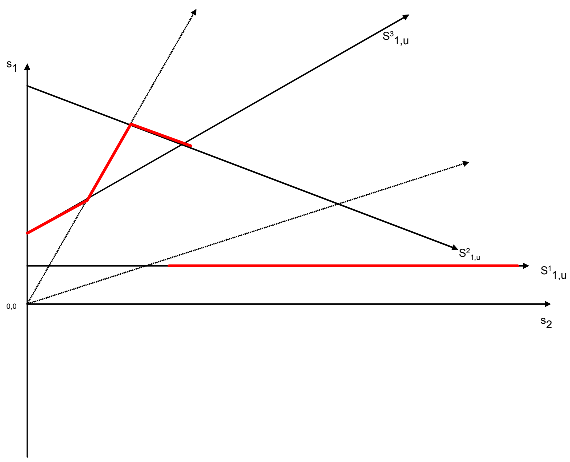

Figure 5: A typical best response correspondence under CEL.

Equating the first derivative to zero, we obtain the unconstrained maximizer

$$s_{i,u}^{2}\left(s_{j}\right)=\frac{2}{a_{i}\left(1-\beta+2r\right)}\ln\left(\frac{2pr}{\left(1-p\right)\left(1-\beta\right)}\right)-\frac{\left(1-\beta\right)}{\left(1-\beta+2r\right)}s_{j}.$$

Therefore, the maximizer s_i^2 of U_i^A in the interval $[\frac{1-\beta}{1+\beta}s_j, \frac{1+\beta}{1-\beta}s_j]$ is $s_i^2\left(s_j\right).$

For the third case, assume $s_i \geq \frac{1+\beta}{1-\beta}s_j$. Then $U_i^L\left(s_i, s_j\right) = -pe^{-a_i r s_i} - (1-p)e^{a_i(1-\beta)s_i - a_i\beta s_j}$. The first derivative is

$$\frac{\partial U_{i}^{L}(s_{i},s_{-i})}{\partial s_{i}}=-\left(1-p\right)\left(1-\beta\right)a_{i}\exp\left(-a_{i}s_{i}\left(\beta-1\right)-\beta a_{i}s_{j}\right)+pra_{i}e^{-ra_{i}s_{i}}$$

and the second derivative is

$$\frac{\partial^{2}U_{i}^{L}(s_{i},s_{-i})}{\partial s_{i}^{2}}=-\left(1-p\right)\left(1-\beta\right)^{2}a_{i}^{2}\exp\left(-a_{i}s_{i}\left(\beta-1\right)-\beta a_{i}s_{j}\right)-pr^{2}a_{i}^{2}e^{-ra_{i}s_{i}}<0.$$

Equating the first derivative to zero, we obtain the unconstrained maximizer

$$s_{i,u}^{3}\left(s_{j}\right)=\frac{1}{a_{i}\left(1+r-\beta\right)}\ln\left(\frac{pr}{\left(1-p\right)\left(1-\beta\right)}\right)+\frac{\beta}{\left(1+r-\beta\right)}s_{j}.$$

Therefore, the maximizer s_i^3 of U_i^A in the interval $[\frac{1+\beta}{1-\beta}s_j, \infty)$ is $s_i^3\left(s_j\right).$ ■

The following lemma characterizes agent i's best response function. See Figure 5 for a typical configuration.

Lemma 12 *For each set of parameter values, there are $K_1, K_2 \in [0,\infty]$ such that $K_1 \leq K_2$ and*

$$BR_i(s_j) = \begin{cases} s_{i,u}^3(s_j) & if \quad 0 \leq s_j < K_1, \\ s_i^2(s_j) & if \quad K_1 \leq s_j < K_2 \\ \{s_i^2(s_j), s_i^1(s_j)\} & if \quad s_j = K_2, \\ s_i^1(s_j) & if \quad K_2 < s_j. \end{cases}$$

Also, $K_2 > \frac{1+\beta}{(1-\beta)(1+r)a_i} \ln\left(\frac{pr}{1-p}\right)$.

Proof. First note that for $s_j = 0$, we have $s_{i,u}^1(0) = \frac{1}{(1+r)a_i} \ln\left(\frac{pr}{1-p}\right)$, $s_{i,u}^2(0) = \frac{2}{a_i(1-\beta+2r)} \ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)$, and $s_{i,u}^3(0) = \frac{1}{a_i(1+r-\beta)} \ln\left(\frac{pr}{(1-p)(1-\beta)}\right)$.Thus, $s_{i,u}^2(0) > s_{i,u}^3(0) > s_{i,u}^1(0)$. There are several cases.

Case 1: $\frac{2pr}{(1-p)(1-\beta)} \leq 1$. Then, $0 \geq s_{i,u}^2(0) > s_{i,u}^3(0) > s_{i,u}^1(0)$. Since $s_{i,u}^2$ has negative slope, $s_i^2(s_j) = \frac{1-\beta}{1+\beta}s_j$ for all s_j. Similarly, since the slope of $s_{i,u}^3$ is $\frac{\beta}{(1+r-\beta)} < \frac{1+\beta}{1-\beta}$, we have $s_i^3(s_j) = \frac{1+\beta}{1-\beta}s_j$ for all s_j. Finally, since $s_{i,u}^1$ is a constant function, $s_{i,u}^1(s_j) < 0$, for all s_j which implies $s_i^1(s_j) = 0$ for all s_j. Thus, $BR_i(s_j) = 0$ for all s_j. Choosing $K_1 = 0$ and $K_2 = \infty$ gives the desired form.

Case 2: $\frac{2pr}{(1-p)(1-\beta)} > 1$, $\frac{pr}{(1-p)(1-\beta)} \leq 1$, and $\ln\frac{pr}{(1-p)(1-\beta)} + \ln(1+\beta) \leq 0$. Then, $s_{i,u}^2(0) > 0 \geq s_{i,u}^3(0) > s_{i,u}^1(0)$. Thus $BR_i(0) = 0$. First note that, for higher s_j, $s_i^1(s_j)$ is a better response than $s_i^2(s_j)$. This is because $U_i^L(s_i^1(0)) = U_i^L(s_i^2(0))$, $U_i^L(s_i^1(.))$ is constant and $U_i^L(s_i^2(.))$ is decreasing in s_j. Therefore, for $s_j > 0$, candidate for best response are $s_i^1(s_j)$ (which is a constant function and thus, gives the same payoff) and $s_i^3(s_j) = \frac{1+\beta}{1-\beta}s_j$. Now note that the derivative of U_i^L evaluated at $s_i = \frac{1+\beta}{1-\beta}s_j$ with respect to s_j is positive if and only if $s_j < (1-\beta)\frac{\ln\frac{pr}{(1-p)(1-\beta)} + \ln(1+\beta)}{a_i(1+r+\beta r-\beta)}$. Due to our assumption $\ln\frac{pr}{(1-p)(1-\beta)} + \ln(1+\beta) \leq 0$ for Case 2, the right hand side of this inequality is nonpositive. Therefore, following $s_i = \frac{1+\beta}{1-\beta}s_j$ decreases agent i's payoff while choosing $s_i = 0$ keeps it constant. Thus in this case, $BR_i(s_j) = 0$ for all s_j. Choosing $K_1 = 0$ and $K_2 = \infty$ gives the desired form.

Case 3: $\frac{2pr}{(1-p)(1-\beta)} > 1$, $\frac{pr}{(1-p)(1-\beta)} \leq 1$, and $\ln\frac{pr}{(1-p)(1-\beta)} + \ln(1+\beta) > 0$. Then, $s_{i,u}^2(0) > 0 \geq s_{i,u}^3(0) > s_{i,u}^1(0)$. Thus $BR_i(0) = 0$. Now the derivative of U_i^L evaluated at $s_i = \frac{1+\beta}{1-\beta}s_j$ with respect to s_j is positive if and only if $s_j < (1-\beta)\frac{\ln\frac{pr}{(1-p)(1-\beta)} + \ln(1+\beta)}{a_i(1+r+\beta r-\beta)}$. Due to our assumption $\ln\frac{pr}{(1-p)(1-\beta)} + \ln(1+\beta) > 0$ for Case 3, the right hand side of this inequality is positive. Therefore, for $s_j < (1-\beta)\frac{\ln\frac{pr}{(1-p)(1-\beta)} + \ln(1+\beta)}{a_i(1+r+\beta r-\beta)}$, following $s_i = \frac{1+\beta}{1-\beta}s_j$ increases agent i's payoff while choosing $s_i = 0$ keeps it constant. For higher s_j, the payoff from $s_i = \frac{1+\beta}{1-\beta}s_j$ starts to decrease and there is a critical value K_2 at which $U_i^L\left(s_i^2\left(K_2\right)\right)=U_i^L\left(s_i^1\left(K_2\right)\right)$. Thus for $0<s_j<K_2$, $BR_i\left(s_j\right)=s_i^2\left(s_j\right)$; at $s_j=K_2$, $BR_i\left(s_j\right)=\left\{s_i^2\left(s_j\right),s_i^1\left(s_j\right)\right\}$; and at $s_j>K_2$, $BR_i\left(s_j\right)=s_i^1\left(s_j\right)$.

Case 4: $\frac{pr}{(1-p)(1-\beta)}>1$. Then, $s_{i,u}^2(0)>s_{i,u}^3(0)>s_i^1(0)\geq 0$. Let s_j^* solve $s_i^3\left(s_j\right)=\frac{1+\beta}{1-\beta}s_j$. (This equality has a solution since the slope of s_i^3 is $\frac{\beta}{(1+r-\beta)}<\frac{1+\beta}{1-\beta}$.) Let s_j^{**} solve $s_i^2\left(s_j\right)=\frac{1+\beta}{1-\beta}s_j$. (This equality has a solution since s_i^2 is downward sloping.) We then have $s_j^*<s_j^{**}$.

Now $BR_i(0)=s_{i,u}^3(0)$ and $U_i^L\left(s_{i,u}^3(.)\right)$ is increasing in s_j (while $U_i^L\left(s_i^1(.)\right)$ is constant). Also, for $s_j\in[0,s_j^*]$, $s_i^2\left(s_j\right)=\frac{1+\beta}{1-\beta}$. Thus, on this interval, $BR_i\left(s_j\right)=s_{i,u}^3\left(s_j\right)$. In the interval $[s_j^*,s_j^{**}]$, $s_i^2\left(s_j\right)=s_i^3\left(s_j\right)=\frac{1+\beta}{1-\beta}$. Note that the payoff from following $\frac{1+\beta}{1-\beta}$ is increasing in s_j as long as $s_j\leq s_j^{***}=(1-\beta)\frac{\ln\frac{pr}{(1-p)(1-\beta)}+\ln(1+\beta)}{a_i(1+r+\beta r-\beta)}$. Now, $s_j^*<s_j^{***}$. Thus, in the interval $[s_j^*,s_j^{***}]$, $BR_i\left(s_j\right)=\frac{1+\beta}{1-\beta}$. As s_j increases beyond s_j^{***}, payoff from $\frac{1+\beta}{1-\beta}$ starts to decrease whereas payoff from $s_i^1\left(s_j\right)$ is constant. Let K_2 be the value of s_j at which $U_i^L\left(s_i^2\left(s_j\right)\right)=U_i^L\left(s_i^1\left(s_j\right)\right)$. Then, in the interval $[s_j^{***},K_2)$, $BR_i\left(s_j\right)=s_i^2\left(s_j\right)$. Also, $BR_i\left(K_2\right)=\left\{s_i^2\left(K_2\right),s_i^1\left(K_2\right)\right\}$ and in the interval (K_2,∞), $BR_i\left(s_j\right)=s_i^1\left(s_j\right)$. This gives us the desired form. ■

Proof. (Proposition 4) Let s^* be a Nash equilibrium of the investment game.

First assume that $BR_i\left(s_j^*\right)=s_i^2\left(s_j^*\right)$ for some $i\in N$. Then, $s_i^2\left(s_j^*\right)=s_{i,u}^2\left(s_j^*\right)$ (since otherwise, $s_i^2\left(s_j^*\right)=\frac{1+\beta}{1-\beta}s_j^*$ and thus, $s_j^*=\frac{1-\beta}{1+\beta}s_i^*$, but then by Lemma 12, $K_2>\frac{1+\beta}{(1-\beta)(1+r)a_j}\ln\left(\frac{pr}{1-p}\right)$ implies that s_j^* can not be a best response). This implies $BR_j\left(s_i^*\right)=s_{j,u}^2\left(s_i^*\right)$. Solving the system

$$
\begin{aligned}
s_i^* &= \frac{2}{a_i\left(1-\beta+2r\right)}\ln\left(\frac{2pr}{\left(1-p\right)\left(1-\beta\right)}\right)-\frac{\left(1-\beta\right)}{\left(1-\beta+2r\right)}s_j^*\\
s_j^* &= \frac{2}{a_j\left(1-\beta+2r\right)}\ln\left(\frac{2pr}{\left(1-p\right)\left(1-\beta\right)}\right)-\frac{\left(1-\beta\right)}{\left(1-\beta+2r\right)}s_i^*
\end{aligned}
$$

we obtain for both $i\in N$,

$$
s_i^*=\frac{\left(2r-\beta+1\right)}{2r\left(r-\beta+1\right)}\left(\frac{1}{a_i}-\frac{\left(1-\beta\right)}{\left(1-\beta+2r\right)}\frac{1}{a_j}\right)\ln\left(\frac{2pr}{\left(1-p\right)\left(1-\beta\right)}\right).
$$

Now assume that $BR_i\left(s_j^*\right)=s_i^1\left(s_j^*\right)$. Then $BR_j\left(s_i^*\right)=s_j^3\left(s_i^*\right)$. If $\ln\left(\frac{pr}{1-p}\right)>0$, $s_i^1\left(s_j^*\right)=s_i^*=\frac{1}{a_i(1+r)}\ln\left(\frac{pr}{1-p}\right)$. Since $s_j^3\left(s_i^*\right)=\frac{1}{a_j(1+r-\beta)}\ln\left(\frac{pr}{(1-p)(1-\beta)}\right)+\frac{\beta}{(1+r-\beta)}s_i^*$, we obtain

$$
s_j^*=\frac{1}{a_j\left(1+r-\beta\right)}\ln\left(\frac{pr}{\left(1-p\right)\left(1-\beta\right)}\right)+\frac{\beta}{\left(1+r-\beta\right)}\frac{1}{a_i\left(1+r\right)}\ln\left(\frac{pr}{1-p}\right)
$$

Alternatively if $\ln\left(\frac{pr}{1-p}\right) < 0$, $s_i^1\left(s_j^*\right) = s_i^* = 0$. This implies

$$s_j^* = \frac{1}{a_j\left(1+r-\beta\right)} \ln\left(\frac{pr}{\left(1-p\right)\left(1-\beta\right)}\right).$$

■