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COSTLY PARTICIPATION**

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# Resale in Second-Price Auctions with Costly Participation\*

Gorkem Celik and Okan Yilankaya<sup>†</sup>

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## Abstract

We investigate efficiency properties of sealed-bid second-price auctions with costly participation and resale. Each bidder chooses to participate in the auction if her valuation is higher than her optimally chosen participation cutoff. If resale is not allowed and the bidder valuations are drawn from a strictly convex distribution function, the symmetric equilibrium (where all bidders use the same cutoff) is less efficient than a class of two-cutoff asymmetric equilibria. Existence of these equilibria without resale is sufficient for existence of similarly constructed two-cutoff equilibria with resale. Moreover, these equilibria with resale are “more asymmetric” and (under a sufficient condition) more efficient than the corresponding equilibria without resale.

*JEL Classification:* C72, D44, D82

*Keywords:* Second-price auctions; resale; participation cost; endogenous entry; endogenous valuations

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# 1 Introduction

We study resale in an independent private values auction setting with costly participation, with a particular focus on efficiency. The seller uses a sealed-bid second-price auction. Bidders are ex-ante symmetric: Their (use) values are drawn from the same distribution function. After learning their private valuations, bidders simultaneously decide whether to participate in the auction or not. Bidders who choose to participate incur a common real resource cost.<sup>1,2</sup>

In the absence of resale opportunities, there is a (unique) symmetric equilibrium of the second-price auction where each bidder bids her valuation iff it is larger than a *participation cutoff* that is common to all bidders. However, there may also be asymmetric equilibria with bidder-specific cutoffs.<sup>3</sup> We first show that, when the valuations are distributed according to a strictly convex cumulative distribution function, there are asymmetric equilibria which are ex-ante more efficient than the symmetric equilibrium. Existence of asymmetric equilibria under strict convexity has been established by Tan and Yilankaya (2006): For any arbitrary partition of the bidders into two groups, there exists an equilibrium where the bidders within a group use the same participation cutoff that differs from the other group's cutoff. We complement this finding by showing that these two-cutoff equilibria provide a higher expected social surplus (net of participation costs) than does the symmetric equilibrium (Proposition 1). The relevance of this result extends beyond second-price auctions, since Stegeman (1996) shows that one of the equilibria of the second-price auction maximizes social surplus within the class of all incentive-compatible allocation rules.<sup>4</sup>

The second-price auction allocates the object to the highest valuation bidder *among participants* in all equilibria where participating bidders bid their values. Yet, when the equilibrium is asymmetric, there is a possibility that a

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<sup>1</sup>Purchasing bid documents, registering or pre-qualifying for the auction, being at the auction site, arranging for financing ahead of time and preparing a bid (which is often a detailed plan with documentation, especially in government procurement) are all costly activities.

<sup>2</sup>This set-up was introduced by Samuelson (1985), and studied by, among others, Stegeman (1996), Campbell (1998), Tan and Yilankaya (2006, 2007) and Celik and Yilankaya (2009). Also see Green and Laffont (1984), where costs as well as valuations are private information.

<sup>3</sup>See Stegeman (1996) for an example, and Tan and Yilankaya (2006) for sufficient and necessary conditions for existence of asymmetric equilibrium.

<sup>4</sup>In Stegeman's (1996) example the asymmetric equilibrium is more efficient than the symmetric one. Celik and Yilankaya (2009) provides a characterization result for efficient auctions.

non-participating bidder has a higher valuation than the winner of the auction. This allocative inefficiency implies that there are potential gains from further trade through resale. Hence we incorporate the possibility of resale (via an optimal auction maximizing the reseller’s revenue) and study its impact on equilibrium behavior and efficiency.

Suppose that there exists a two-cutoff asymmetric equilibrium of the second-price auction without resale, where one group (aggressive participants) has a low cutoff and the other group has a higher one. We show that there also exists an equilibrium that partitions bidders the same way when resale is allowed. Moreover, this resale equilibrium is “more asymmetric” than the corresponding no-resale equilibrium: The low cutoff decreases and the high cutoff increases (Proposition 2). The prospect of reselling the good induces the low-cutoff bidders to enter even more aggressively and the possibility of buying the object in the resale phase makes the high-cutoff bidders even more hesitant to enter. Moreover, there is overbidding by low-cutoff bidder types who hope to resell: They bid their adjusted values (expected payoffs inclusive of the resale phase), which are higher than their use values.

Fixing participation and bidding behavior in the initial auction, resale enhances efficiency as the object is potentially transferred to a higher-value bidder. However, resale may also affect the equilibrium cutoffs and bids. Nevertheless, provided that a sufficient condition is satisfied, allowing resale improves ex-ante efficiency: Whenever there is a two-cutoff asymmetric equilibrium without resale, the corresponding more asymmetric equilibrium with resale yields a higher social surplus (Proposition 3). This sufficient condition is on the distribution of valuations and hence independent of the magnitude of the participation cost and the number of bidders.

Resale is commonly observed after auctions in many markets. There are a few sources of gains to resale trade offered in the literature. New bidders or more information to existing bidders may arrive between the initial auction and the resale stage (Bikchandani and Huang, 1989; Haile, 1996 and 2003; Bose and Deltas, 2006). Bidder asymmetries may also cause inefficiencies in first-price auctions (Gupta and Lebrun, 1999; Hafalir and Krishna, 2008; Cheng and Tan, 2010; Lebrun, 2010a; Virág, 2013).<sup>5</sup>

In second-price and English auctions, even when resale is allowed, (use) value-bidding remains to be an equilibrium (see, for example, Haile, 1996). This equilibrium allocates the object to the highest value bidder and hence there is no resale on the path of play. However, bidding one’s value is no longer weakly dominant when resale is allowed. Garratt and Tröger (2006)

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<sup>5</sup>Zheng (2002) studies the optimal auction with resale. Also see Lebrun (2012).

identify alternative equilibria where even a speculator with no use value can make positive profits. Garratt, Tröger, and Zheng (2009) construct equilibria with a designated bidder (potential reseller) which can then be used to support collusion (by rotating the designated bidder). As we discussed above, when there are participation costs, second-price auctions may have asymmetric equilibria in undominated strategies, where all *participants* bid their values. We investigate the resale opportunities naturally arising from these equilibria. In our equilibria, *all* types of *all* participants bid their adjusted values that reflect potential gains from the resale auction.

The closest paper to ours is by Xu, Levin, and Ye (2013), who study second-price auctions with resale, where valuations and participation costs are both private information. They show that a symmetric equilibrium exists and is unique under some conditions. Participants in the initial auction bid their adjusted values described above and there is resale in equilibrium. Resale opportunities arise because of differences in participation costs: When a low-cost bidder wins the object, she can resell it to a high-cost bidder with a higher valuation (who did not participate in the initial auction). Further analytical results are difficult to obtain with two-dimensional private information. Their numerical analysis suggests that the effect of resale on efficiency (and on revenue) is ambiguous. In our model with commonly known participation costs, we show that heterogeneity of costs is not necessary for equilibrium resale. Instead, resale opportunities are generated by asymmetric equilibria. This setting also allows us to obtain an analytical result on the impact of resale on efficiency.

In the next section, we describe the environment and study the benchmark case, where resale is not allowed. We study resale in Section 3, analyzing the optimal resale auction, and the participation and bidding behavior in the initial auction. We provide some concluding remarks in Section 4. All proofs are in the Appendix.

## 2 No Resale

We consider a symmetric independent private values environment. There is a risk-neutral seller who owns an indivisible object and is selling it via a sealed-bid second-price auction without a reserve price. Her valuation is normalized to be 0. There are  $n \geq 2$  risk-neutral (potential) bidders. Let  $v_i$  denote the (use) value of bidder  $i \in \{1, \dots, n\}$  for the object. Bidders' valuations are independently distributed according to the cumulative distribution function (cdf)  $F$  on  $[0, 1]$ , with continuously differentiable and positive density function

$f$ . Bidders know their own valuations. For simplicity, we assume that a generalized version of the monotone hazard rate condition is satisfied: For any  $x \in (0, 1]$ ,  $\frac{F(x)-F(v)}{f(v)}$  is strictly decreasing in  $v$  for all  $v \in [0, x]$ . Note that this condition automatically holds for convex distribution functions (for weakly increasing density functions  $f$ ).

There is a participation cost, common to all bidders, denoted by  $c \in (0, 1)$ : Bidders must incur this real resource cost in order to be able to submit a bid.<sup>6</sup> All bidders make their participation and bidding decisions simultaneously. They know their valuations when making these decisions.

## 2.1 Equilibrium

We first study the benchmark case where there is no resale possibility. If a bidder decides to participate in the second-price auction, she cannot do better than bidding her own valuation. Accordingly, we only consider (Bayesian-Nash) equilibria in *cutoff strategies*: Each bidder bids her valuation if it is greater than a cutoff point and does not participate otherwise.<sup>7</sup> However, there may be asymmetric equilibria where bidders have different participation cutoffs. In what follows, we restrict attention to equilibria with (at most) two cutoffs. Since our results are of the existence/possibility variety, this restriction has no bearing on them, while simplifying the exposition considerably.<sup>8</sup>

Suppose that  $l$  bidders use the low cutoff  $a$  and  $h = n - l$  bidders use the high cutoff  $b$  in some equilibrium, with  $0 \leq a \leq b \leq 1$  and  $1 \leq l \leq n$ . These cutoffs are determined by indifference (to participation) conditions. To find them, first consider the participation decision of one of the  $l$  bidders who has the lower cutoff  $a$  and whose valuation is also  $a$ . Suppose that all other bidders are following their equilibrium strategies. She obtains the object iff she is the only bidder to participate, which happens with probability  $F(a)^{l-1} F(b)^h$ , and hence pays 0 if she wins. Her expected payoff from participation is then

$$F(a)^{l-1} F(b)^h a - c. \tag{1}$$

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<sup>6</sup>For our positive results about equilibria in auctions with or without resale,  $c$  can also be interpreted as an entry fee (charged by the seller).

<sup>7</sup>We adopt the convention that if a bidder's cutoff is 0 (respectively, 1) all (respectively, none of) her types are participating. The participation decisions of these zero measure cutoff types are inconsequential.

<sup>8</sup>Tan and Yilankaya (2006) show that, for *any*  $c$ , log-concavity of  $F(\cdot)$  is sufficient for the upper limit of distinct cutoffs to be two in any (cutoff) equilibria.

Similarly, the expected payoff of a high-cutoff bidder with valuation  $b$  is given by

$$\begin{aligned} & F(a)^l F(b)^{h-1} b + F(b)^{h-1} \int_a^b (b-w) dF(w)^l - c \\ = & F(b)^{h-1} [F(a)^l a + \int_a^b F(w)^l dw] - c. \end{aligned} \quad (2)$$

Define the following functions:  $\tilde{\pi}_L(a, b) = F(a)^{l-1} F(b)^h a$  and  $\tilde{\pi}_H(a, b) = F(b)^{h-1} [F(a)^l a + \int_a^b F(w)^l dw]$ .

For  $a^* \leq b^*$  to be equilibrium cutoffs, the following conditions must be satisfied:

$$\begin{aligned} \tilde{\pi}_L(a^*, b^*) & \geq c, \text{ with equality if } a^* > 0, \\ \tilde{\pi}_H(a^*, b^*) & \leq c, \text{ with equality if } b^* < 1. \end{aligned} \quad (3)$$

Any bidder with a value lower than  $c$  will have a negative payoff from participation. So, we know that  $a^* \geq c > 0$ , and the first condition will be satisfied with equality.<sup>9</sup>

We note the following observation, which will be used later:

**Remark 1**  $\tilde{\pi}_L(a, b)$  and  $\tilde{\pi}_H(a, b)$  are strictly increasing in  $a$  and  $b$  for  $a > 0$ . ( $\tilde{\pi}_L(0, b) = 0 \forall b$  and  $\tilde{\pi}_H(0, b)$  is strictly increasing in  $b$ .)

Notice that, since we allow for the possibility that  $a = b$ , the symmetric equilibrium is a special case within the class of (at most) two cutoff equilibria. There always exists a symmetric equilibrium where all bidders use the same participation cutoff  $a = b = v_s \in (c, 1)$ , where

$$F(v_s)^{n-1} v_s = c. \quad (4)$$

Tan and Yilankaya (2006) show that strict convexity of  $F$  is sufficient for existence of two-cutoff asymmetric equilibria for *any*  $l$ , the number of bidders using the lower cutoff. It may be helpful to go over their argument with graphs, which we will also utilize for our results. Consider the set  $\Delta = \{(a, b) : 0 \leq a \leq b \leq 1\}$  in  $R^2$  that identifies feasible participation cutoff pairs, and the curves given by  $\tilde{\pi}_L(a, b) = c$  and  $\tilde{\pi}_H(a, b) = c$  in  $\Delta$ . When  $F$  is strictly convex, the second curve is steeper than the first one at  $(v_s, v_s)$ ,

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<sup>9</sup>We keep the conditions in their current forms to make them directly comparable to the corresponding conditions in the resale case, where a bidder with valuation less than  $c$  may participate in order to sell later.

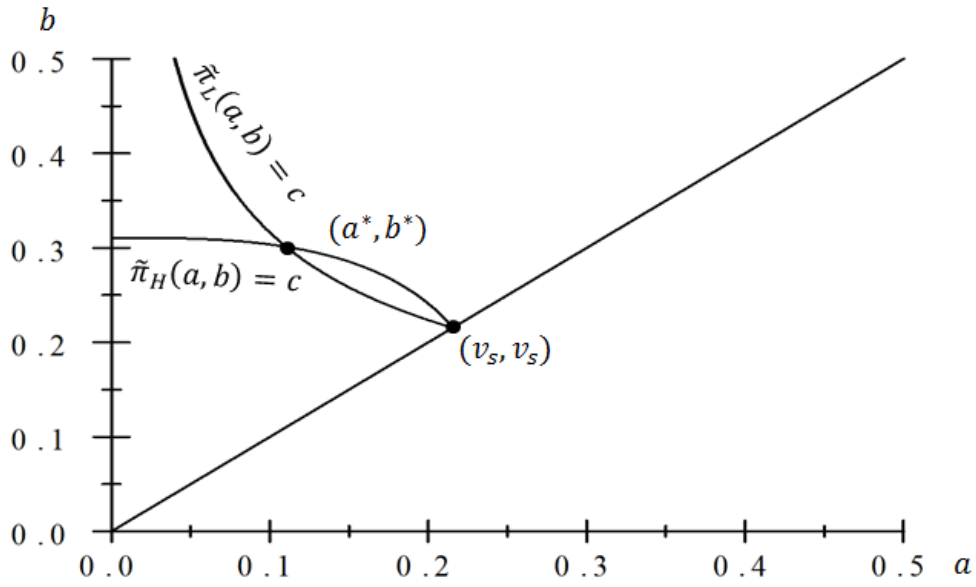


Figure 1:  $F(v) = v^2, n = 2, c = 0.01$ . Symmetric equilibrium with  $v_s = 0.215$ . Asymmetric equilibrium with  $a^* = 0.11, b^* = 0.301$ .

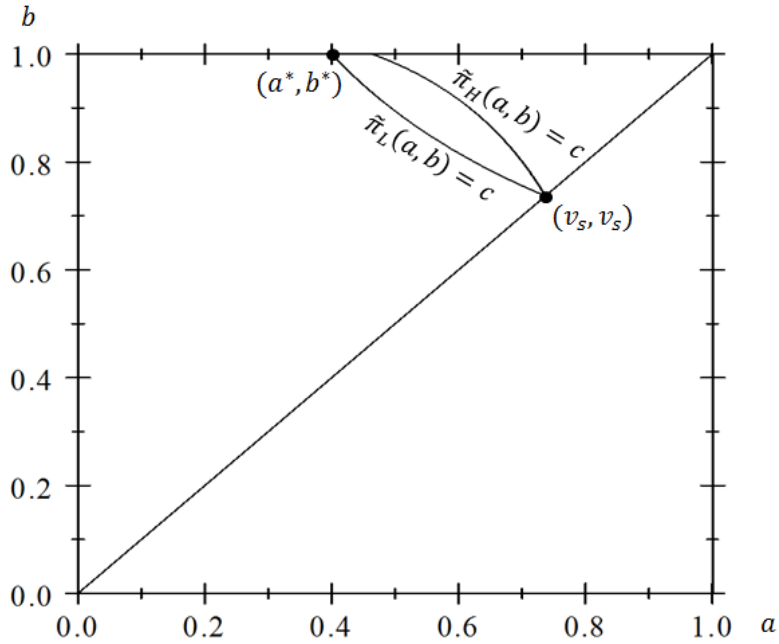


Figure 2:  $F(v) = v^2, n = 2, c = 0.4$ . Symmetric equilibrium with  $v_s = 0.737$ . Asymmetric equilibrium with  $a^* = 0.4, b^* = 1$ .



the symmetric equilibrium cutoffs. If these curves intersect in the interior of  $\Delta$ , their intersection yields the cutoffs for an asymmetric equilibrium, satisfying (3) with equalities (Figure 1). Otherwise, we have a corner asymmetric equilibrium with  $a^* \in [c, v_s)$  and  $b^* = 1$ , where  $\tilde{\pi}_L(a^*, 1) = c$  and  $\tilde{\pi}_H(a^*, 1) \leq c$ , as depicted in Figure 2.

## 2.2 Efficiency

We can write down the social surplus as a function of the two cutoff points in the no-resale setting:

$$\tilde{S}(a, b) = \int_a^b v F(b)^h dF(v)^l + \int_b^1 v dF(v)^{h+l} - l(1 - F(a))c - h(1 - F(b))c. \quad (5)$$

The first integral measures the expected value of the object for the winner of the auction when she is a low cutoff bidder with a value on interval  $[a, b]$ , and the second one is the expected value for a winner with a valuation higher than  $b$ . The last two terms are expected participation costs incurred by all bidders. Note that the seller's valuation is normalized to be 0 and the payment made by the winning bidder is just a transfer to the seller.

The derivatives of this social surplus function with respect to its two arguments can be written by referring to functions  $\tilde{\pi}_L$  and  $\tilde{\pi}_H$  that we just defined:

$$\begin{aligned} \frac{\partial \tilde{S}(a, b)}{\partial a} &= -lf(a) [\tilde{\pi}_L(a, b) - c], \\ \frac{\partial \tilde{S}(a, b)}{\partial b} &= -hf(b) [\tilde{\pi}_H(a, b) - c]. \end{aligned} \quad (6)$$

These first order conditions imply that the social surplus is decreasing in  $a$  and increasing in  $b$  for the set of points where  $\tilde{\pi}_H(a, b) \leq c \leq \tilde{\pi}_L(a, b)$ , i.e., the lens-shaped areas in Figures 1 and 2. Accordingly, when  $F$  is convex, social surplus will be higher on the asymmetric equilibria identified by Tan and Yilankaya (2006) in comparison to the symmetric equilibrium.

**Proposition 1** *If  $F$  is strictly convex, then, for any  $l \in \{1, 2, \dots, n - 1\}$ , there exists an asymmetric no-resale equilibrium where  $l$  bidders use one cutoff and  $h = n - l$  bidders use another one that generates a higher social surplus than the symmetric equilibrium.*

This result holds regardless of the magnitude of the participation cost  $c$ , the number of bidders  $n$ , and the way bidders are classified into low and high-cutoff groups. For given levels of  $c$  and  $n$ , strict convexity of  $F$  in Proposition 1

can be replaced with the weaker local condition that  $\frac{F(v)}{v}$  is strictly increasing at the symmetric cutoff  $v_s$ , defined in (4). This local condition is all that is needed to generate the lens-shaped areas such as those in Figures 1 and 2. Strict convexity implies that  $\frac{F(v)}{v}$  is strictly increasing for all  $v$ . Finally, note that the results identifies at least  $n - 1$  distinct (ignoring permutations of bidder identities) asymmetric equilibria of the second-price auction, and each of them has a higher surplus than the symmetric equilibrium.<sup>10</sup>

### 3 Resale

Asymmetric equilibria of the second-price auction with participation costs, such as those we discussed above, have the following feature: Even though the object is obtained by the bidder who has the highest valuation *among participants*, a nonparticipant may have a higher valuation. Therefore, when the winner of the auction has a valuation which is lower than the participation cutoff of another bidder, there are potential gains from further trade. To investigate this issue, we now allow for a resale stage where the winner of the initial auction can design her own resale auction for potential bidders.

Timing is as follows: Bidders make participation and bidding decisions simultaneously in the initial auction. The winner designs a resale auction if she chooses to do so. Others make their participation and bidding decisions in this resale auction.

We assume that the highest bidder learns that she is the winner and does not learn others' bids. We make this no-disclosure assumption only for notational simplicity. Full-disclosure of all bids (or any other intermediate disclosure policy) would not affect our equilibrium outcome.<sup>11</sup> We assume that there are no participation costs in the resale stage, and discuss this assumption in our concluding remarks.

We look for (Perfect Bayesian) equilibria of this game where bidders are divided into two groups that use two (potentially distinct) participation cutoffs

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<sup>10</sup>Since the second-price auction has an (ex-ante) efficient equilibrium in this setting (Stegeman, 1996), Proposition 1 implies that the efficient auction is asymmetric for strictly convex distribution functions. There is a connection between optimal (revenue-maximizing) and efficient auctions. Following Myerson (1981), define  $J(v) = v - \frac{1-F(v)}{f(v)}$  as the *virtual value* of a bidder with value  $v$ . In Celik and Yilankaya (2009), we showed that, if  $\frac{F(v)}{J(v)}$  is strictly increasing, then the optimal auction is asymmetric, implying (using the connection we mentioned) that the efficient auction is asymmetric when  $\frac{F(v)}{v}$  is strictly increasing.

<sup>11</sup>See Lebrun (2010b) for a discussion of importance of bid disclosure policies, especially in first-price auctions with resale.

in the initial auction, just like before. Similarly, we restrict attention to natural equilibria in which participants in the initial auction bid their adjusted values (gross expected payoff inclusive of the resale stage).<sup>12</sup>

We analyze the optimal resale auction first (given the restrictions above), followed by bids and equilibrium participation cutoffs in the initial auction.

### 3.1 Optimal resale auction

Suppose that  $l$  bidders use cutoff  $a$  and  $h = n - l$  bidders use cutoff  $b$  in the initial auction, with  $0 \leq a \leq b \leq 1$  and  $1 \leq l \leq n$ . Suppose further that bids are monotone increasing in valuations and that bidders with identical valuations bid the same amount (if they participate). In this equilibrium we are constructing, there are opportunities for resale only if a bidder wins the initial auction with a value between  $a$  and  $b$ . The bidders who are using the higher cutoff  $b$  in the initial auction are the potential buyers in the resale stage. The winner of the initial auction (one of the  $l$  bidders who use  $a$  as the cutoff and who has a valuation in  $[a, b]$ ) has learned that none of these high-cutoff bidders have a value higher than  $b$ , otherwise they would have participated in the initial auction and acquired the good. Therefore, the problem she is facing is finding the optimal auction for  $h \geq 1$  bidders whose valuations are independently distributed on  $[0, b]$  according to the cdf  $\frac{F(\cdot)}{F(b)}$ .<sup>13,14</sup>

If the reseller's valuation were commonly known, this would be the standard optimal auction problem à la Myerson (1981). However, it is not known, and so we have an “informed principal” problem. Fortunately, it is possible to show that this does not matter in this independent private values setting. It

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<sup>12</sup>Note that bidding this “adjusted value” is no longer the dominant strategy even conditional on participating, since this value is calculated using the equilibrium expected payoff from the resale auction. However, given an equilibrium of the resale auction (which itself will be in dominant strategies for the resale stage), bidding the adjusted value will indeed be dominant conditional on participation, as usual, explaining our use of the word “natural.”

<sup>13</sup>The generalized monotone hazard rate condition we assumed ensures that we are in the “regular case” with increasing virtual valuations. Notice also that when  $h = 0$  (or  $a = b$ ), we have symmetric participation in the initial auction, and hence there is no room for resale.

<sup>14</sup>We are describing the resale stage only on the equilibrium path (or rather when the initial auction behavior is described by two participation cutoffs and monotone bid functions). We do not formally define the full strategies and the beliefs at the resale stage to keep the exposition simple. There are many resale-stage beliefs that would support the equilibria we are describing, including the passive one where the reseller does not update when she sees an off-the-equilibrium-path bid.

is optimal for each type of the reseller in  $[a, b]$  to choose a standard optimal auction for that type.<sup>15</sup>

There are many auctions which are expected-payoff equivalent for the reseller and the bidders (the revenue equivalence theorem), but we will focus on a second-price auction with an optimal reserve price  $r(w)$  for the reseller with valuation  $w \in [a, b]$ , satisfying<sup>16,17</sup>

$$r = w + \frac{F(b) - F(r)}{f(r)}. \quad (7)$$

The monotone hazard rate condition implies that, for any  $b$ , there is a unique value for  $r(w) \in (w, b]$ . Note that we have  $r'(w) \in (0, 1)$  and  $r(b) = b$ .

A bidder with value  $v$  participates in the resale auction if  $v \geq r$ , and bids  $v$ . It is straightforward to calculate the expected payment she makes to the reseller if she wins the resale auction:<sup>18</sup>

$$\alpha(v, r) = v - \int_r^v \frac{F(x)^{h-1}}{F(v)^{h-1}} dx. \quad (8)$$

### 3.2 Equilibrium bids in the initial auction

Now that we discussed the optimal resale auction on the equilibrium path, we are ready to study bidding in the initial auction. As we mentioned above, we look for an equilibrium in which bids are given by gross expected payoffs, taking the resale stage into account. When the winner is a bidder with a valuation higher than the high cutoff  $b$ , there is no room for resale and the winner's payoff is equal to her (use) value. Therefore such a bidder would bid her value in the initial auction. On the other hand, when the winner is a low-cutoff

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<sup>15</sup>Yilankaya (1999) shows this in the bargaining context, i.e., when  $h = 1$ . The same argument applies for  $h > 1$  (Yilankaya, 2004, available from the authors upon request): The Myerson auction is optimal when the seller's valuation is common knowledge. It is also the seller's ex-ante optimal mechanism. Myerson's *principle of inscrutability* (1983) implies that it will be the informed principal's optimal mechanism. Also see the discussion in Mylovanov and Troger (2014) and Maskin and Tirole (1990).

<sup>16</sup>On the equilibrium path, of course, it does not matter which of the optimal auctions is used in the resale stage. However, it may matter when considering potential deviations in the initial auction.

<sup>17</sup>We suppress the dependence of  $r$  on  $b$  for notational simplicity.

<sup>18</sup>This expression may be familiar as the equilibrium bid function in a first price auction with  $h$  bidders whose valuations are distributed according to  $\frac{F(\cdot)}{F(b)}$  on  $[0, b]$ . Revenue equivalence theorem implies that this is the expected payment of the winner in the second-price auction.

bidder with a valuation  $v \in [a, b]$ , her gross payoff is equal to the expected continuation payoff in the resale stage. In the Appendix (Lemma 1), we show that this continuation payoff is  $b - \int_v^b \frac{F(r(x))^h}{F(b)^h} dx$ , where  $r(x)$  is the optimal reserve price for a bidder with valuation  $x$ , by using the revenue equivalence theorem. Accordingly, the equilibrium bids of the low-cutoff bidders in the initial auction are

$$\beta(v) = \begin{cases} No & \text{if } v < a \\ b - \int_v^b \frac{F(r(x))^h}{F(b)^h} dx & \text{if } a \leq v < b \\ v & \text{if } b \leq v \end{cases}, \quad (9)$$

where “*No*” denotes not participating. Since high-cutoff bidders’ participation precludes a resale stage, their bids will reflect only their use values:

$$\hat{\beta}(v) = \begin{cases} No & \text{if } v < b \\ v & \text{if } b \leq v \end{cases}. \quad (10)$$

The equilibrium bid of a low-cutoff bidder is higher than her use value when it is in  $[a, b)$ . Bidders with such valuations are aware of the possibility that they can resell the good to a high-cutoff bidder who has a higher use value. Since this possibility is decreasing in the valuation of the low-cutoff bidder, the extent of overbidding is decreasing in  $v$  (and it is eliminated for  $v \geq b$ ).

Bidding functions  $\beta(\cdot)$  and  $\hat{\beta}(\cdot)$  imply that the initial auction is ex-post efficient in a constrained sense: It allocates the good to the bidder with the highest valuation among participants. Any inefficiency in the initial auction (therefore, any resale opportunity) is due to possible asymmetric participation behavior, which we discuss next.

### 3.3 Equilibrium participation in the initial auction

A bidder’s participation decision in the initial auction will depend on the comparison of the participation cost with the payoff differential generated by her participation, taking into account the resale stage. We show in the Appendix (Lemma 2) that, for each bidder, this payoff differential is (weakly) increasing in her valuation when other bidders are using cutoff strategies. Therefore, it is sufficient to consider participation incentives for bidders whose valuations are equal their respective cutoffs.

Consider one of the  $l$  low-cutoff bidders who has a valuation equal to her cutoff  $a$ . This bidder cannot buy the object in a resale auction since all the

equilibrium reserve prices will be higher than  $a$ . When she enters in the initial auction, she would be the winner if she is the only participant. Given the other bidders' participation decisions, the probability of this happening is  $F(a)^{l-1} F(b)^h$ . This sole participant does not make any payment, and her expected payoff in the continuation game is  $\beta(a)$ . Therefore, her payoff differential for participation in the initial auction is

$$\pi_L(a, b) = F(a)^{l-1} F(b)^h \beta(a). \quad (11)$$

Now consider one of the  $h$  high-cutoff bidders who has a valuation equal to her cutoff  $b$ . If she participates (and bids  $b$ ), then her expected payoff will be

$$F(b)^{h-1} [F(a)^l b + \int_a^b [b - \beta(w)] dF(w)^l]. \quad (12)$$

To see this, notice that she wins only if none of the  $h - 1$  high-cutoff bidders participates and all the  $l$  low-cutoff bidders have valuations less than  $b$ . She pays 0 if none of the low-cutoff bidders participates (when all of them have valuations less than  $a$ ). Otherwise she pays the bid of the highest-valuation low-cutoff bidder,  $\beta(w)$ .<sup>19</sup>

On the other hand, if she stays out, then her expected payoff will be

$$F(b)^{h-1} \int_a^b [b - \alpha(b, r(w))] dF(w)^l, \quad (13)$$

since resale auction occurs if none of the  $h - 1$  other high-cutoff bidders participates in the initial auction and the highest valuation among low-cutoff bidders is in  $[a, b]$ . The bidder who has this highest valuation  $w$  sets the reserve price  $r(w)$ , so the expected payment of type- $b$  bidder is  $\alpha(b, r(w))$  (see (8)).

Therefore, the payoff differential for a high-cutoff bidder with valuation  $b$  is

$$\pi_H(a, b) = F(b)^{h-1} [F(a)^l b + \int_a^b [\alpha(b, r(w)) - \beta(w)] dF(w)^l]. \quad (14)$$

We prove the following in the Appendix.

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<sup>19</sup>There will not be a resale stage if she participates, since resale happens only if a low-cutoff bidder wins the object with valuation in  $[a, b]$ , and these bidder types bid less than  $b$ , i.e.,  $\beta(w) < b$  for  $w < b$ .

**Remark 2**  $\pi_L(a, b)$  is increasing in  $a$  and  $b$  for  $a > 0$ , and  $\pi_H(a, b)$  is increasing in  $a$ .

We are finally ready to identify the conditions that equilibrium cutoffs  $a^{**}$  and  $b^{**}$  must satisfy, analogous to conditions (3) in the no-resale setting:

$$\begin{aligned} \pi_L(a^{**}, b^{**}) &\geq c, \text{ with equality if } a^{**} > 0, \\ \pi_H(a^{**}, b^{**}) &\leq c, \text{ with equality if } b^{**} < 1. \end{aligned} \tag{15}$$

These conditions admit a symmetric solution, with  $a^{**} = b^{**} = v_s$ , the symmetric equilibrium cutoff of the benchmark no-resale case. There is no resale in this equilibrium, since the bidder with the highest valuation receives the object.

Our next result is about the existence and properties of equilibria with asymmetric cutoffs, where resale is an equilibrium phenomenon. Whenever there is an asymmetric equilibrium (with two cutoffs) in the benchmark no-resale case, there will also be an asymmetric equilibrium with resale. Moreover, the equilibrium with resale will be “more asymmetric.”

**Proposition 2** *Suppose that there exists an asymmetric equilibrium in the benchmark case of no-resale with  $l \in \{1, 2, \dots, n - 1\}$  bidders using the cutoff  $a^*$  and  $h = n - l$  bidders using the cutoff  $b^* > a^*$ . Then there exists an asymmetric equilibrium with resale, where  $l$  bidders use cutoff  $a^{**}$  and  $h$  bidders use cutoff  $b^{**} > a^{**}$ . Moreover,  $a^{**} < a^*$  and  $b^{**} \geq b^*$  (with strict inequality if  $b^* < 1$ ).*

A key step in our proof is showing that  $\pi_L(a, b) > \tilde{\pi}_L(a, b)$  and  $\pi_H(a, b) < \tilde{\pi}_H(a, b)$  for  $b > a > 0$ . These inequalities imply that the asymmetric no-resale equilibrium cutoffs  $(a^*, b^*)$  lie below the curve given by  $\pi_H(a, b) = c$  and above the curve  $\pi_L(a, b) = c$ . The remainder of the argument is very similar to our discussion of asymmetric equilibria in the no-resale benchmark. If these two curves intersect in the interior of  $\Delta = \{(a, b) : 0 \leq a \leq b \leq 1\}$ , their intersection yields the cutoffs for an asymmetric equilibrium with resale, satisfying (15) with equalities. Otherwise, we have a corner equilibrium either with  $a^{**} = 0$  (Figure 3) or with  $b^{**} = 1$  (Figure 4). The resulting equilibrium cutoffs  $(a^{**}, b^{**})$  are more asymmetric than the corresponding no-resale equilibrium cutoffs  $(a^*, b^*)$  in the sense that they are further away from the symmetric equilibrium cutoff  $v_s$ .

In an equilibrium such as the one described above, a low-cutoff bidder participates and bids more aggressively in the initial auction due to the opportunity to resell to a high-cutoff bidder. This opportunity in turn is supported

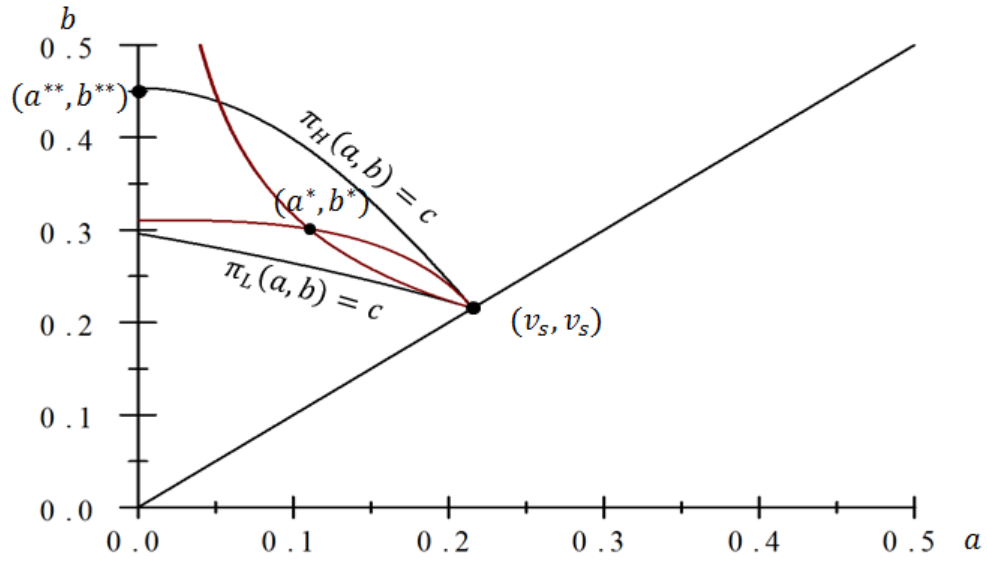


Figure 3:  $F(v) = v^2, n = 2, c = 0.01$ . Symmetric equilibrium with  $v_s = 0.215$ . Asymmetric equilibrium with  $a^{**} = 0, b^{**} = 0.454$ .

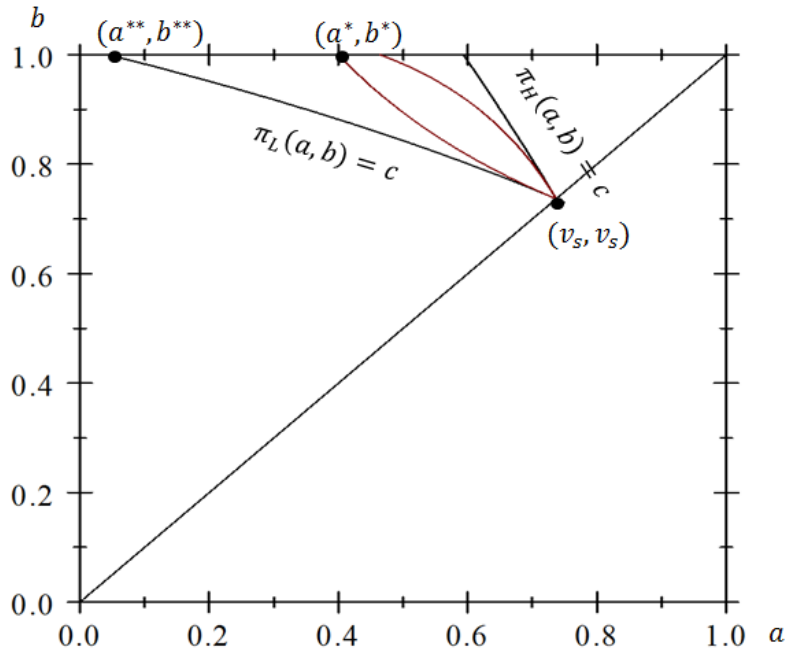


Figure 4:  $F(v) = v^2, n = 2, c = 0.4$ . Symmetric equilibrium with  $v_s = 0.737$ . Asymmetric equilibrium with  $a^{**} = 0.044, b^{**} = 1$ .



by some types of the high-cutoff bidder remaining out of the initial auction to buy later in the resale stage. This asymmetry in behavior arises as an equilibrium phenomenon even though bidders are ex-ante symmetric, as it is the case in the no-resale benchmark. A similar speculative motive also appears in the symmetric equilibrium of Xu, Levin and Ye's (2013) model with two possible (privately-known) participation costs: Bidders with high cost tend to stay out of the initial auction and the low-cost bidders over-enter and over-bid with the hope of reselling the object. With Proposition 2 we show that this speculative motive and resale can arise in equilibrium even when all bidders have the same participation cost.

### 3.4 Efficiency with resale

To examine the welfare effects of resale, we consider the social surplus as a function of two participation cutoffs. This surplus function is constructed under the assumption that, once the bidders enter in or stay out of the initial auction according to these participation cutoffs, they follow the equilibrium bidding and resale strategies described above.

$$\begin{aligned}
 S(a, b) = & \int_a^b \left[ F(r(w))^h w + \int_{r(w)}^b v dF(v)^h \right] dF(w)^l \\
 & + \int_b^1 w dF(w)^{h+l} - l(1 - F(a))c - h(1 - F(b))c.
 \end{aligned} \tag{16}$$

The first integral term refers to the expected surplus if the initial auction allocates the good to a low-cutoff bidder with valuation in  $[a, b]$ . This expected surplus is calculated by taking the possibility of resale into account. The second integral term is the expected surplus when the highest valuation among all bidders is higher than cutoff  $b$ . The last two terms measure the expected cost of participation.

For fixed cutoffs, the possibility of resale increases total welfare:  $S(a, b) > \tilde{S}(a, b)$  for  $a < b$ , since

$$S(a, b) - \tilde{S}(a, b) = \int_a^b \int_{r(w)}^b (v - w) dF(v)^h dF(w)^l. \tag{17}$$

This difference is simply the surplus gain of transferring the object from a low-cutoff bidder with value  $w$  to a high-cutoff bidder with a higher value  $v$  in the resale phase.

Consider an equilibrium in the benchmark case of no-resale with asymmetric cutoffs  $a^*$  and  $b^*$ . As we just observed, if bidders were to use the

same cutoffs when resale is allowed, then the surplus would be higher, i.e.,  $S(a^*, b^*) > \tilde{S}(a^*, b^*)$ . However, the possibility of resale may also change equilibrium participation behavior of the bidders. Therefore, we need to know how the value of function  $S(a, b)$  changes as we move from the no-resale equilibrium cutoffs  $(a^*, b^*)$  to resale equilibrium cutoffs  $(a^{**}, b^{**})$ . With our next result, we provide a sufficient condition for the social surplus to increase when resale is allowed.

**Proposition 3** *Suppose that there exists a two-cutoff asymmetric equilibrium in the benchmark case of no-resale and that  $\frac{vf(v)}{F(v)}$  is weakly increasing. Then there exists an asymmetric equilibrium with resale which generates a higher social surplus than does this asymmetric no-resale equilibrium.*

To prove the proposition, we first show that the social surplus function  $S(a, b)$  is decreasing in  $a$  and increasing in  $b$  whenever  $\pi_H(a, b) < c < \pi_L(a, b)$ , if  $\frac{vf(v)}{F(v)}$  is weakly increasing. The result then follows from inequalities  $a^{**} < a^*$  and  $b^{**} \geq b^*$ .

Proposition 3 provides a sufficient condition on the distribution of valuations only. Hence, it is independent of the magnitude of the participation cost  $c$ , the number of bidders  $n$ , and how these bidders are divided into two groups. The condition is satisfied when the cdf for the valuations is a power function, i.e.,  $F(v) = v^\alpha$  for  $\alpha > 0$  (since  $\frac{vf(v)}{F(v)} = \alpha$ ).

## 4 Concluding Remarks

We study resale and show that it can be an equilibrium phenomenon in a symmetric second-price auction with costly participation. The equilibrium with resale is more asymmetric than the corresponding one without resale due to speculative motives. When resale is not allowed, we identify asymmetric equilibria that are more efficient than the symmetric one if the cdf of bidders' valuations is strictly convex. We provide a sufficient condition for resale to improve (ex-ante) efficiency. Therefore, when  $F$  is strictly convex and this sufficient condition is satisfied, we have a ranking: Symmetric equilibrium (resale allowed or not) is less efficient than the asymmetric no-resale equilibria we identified, which in turn are less efficient than the corresponding asymmetric resale equilibria.

The effect of resale on expected revenue is harder to ascertain. Opportunity to resell the object induces higher participation and higher bids by some of the bidders, leading to a positive impact on revenue. Other bidders, however, would participate less since they might have the option to buy the object

later, identifying a countervailing factor. Therefore the net effect of resale on revenue is ambiguous.<sup>20</sup> There is a similar ambiguity for the effect of resale on the expected number of participants to the auction.

We assume that there are no participation costs at the resale stage. One possible justification is that the reseller may follow a bidder qualification procedure which is less stringent than that of the original seller, e.g., due to the original seller being a public entity (see Xu, Levin, and Ye, 2013, who also assume costless resale). However, the main reason for our assumption is to keep the analysis simple. When there are participation costs, the optimal auction at the resale stage would be more complicated than a standard auction; in particular, it could be asymmetric (Celik and Yilankaya, 2009). These complications aside, we expect the resale equilibria and their main structure to remain intact in the presence of costly resale.

In our model, bidders make their participation and bidding decisions simultaneously. Another possibility is bidders making their participation decisions first and then bidding after having observed the number of participants (e.g., Xu, Levin, and Ye, 2013). This alternative scenario (or even observing the identities of participants) would not change our results, since in equilibrium all bidder types bid their adjusted values inclusive of the payoff from resale stage.

## 5 Appendix

### Proof of Proposition 1 (no-resale)

Following the proof of Proposition 3i in Tan and Yilankaya (2006), for all  $b \in [v_s, 1]$ , define  $\Phi(b)$  implicitly as the value of  $a$  which solves  $\tilde{\pi}_L(a, b) = c$ . Notice that function  $\Phi(b)$  is continuously differentiable, strictly decreasing, and that  $\Phi(v_s) = v_s$ . For all  $b \in [v_s, 1]$ , also define

$$g(b) = \tilde{\pi}_H(\Phi(b), b) - c.$$

This function is also continuously differentiable with  $g(v_s) = 0$ . According to the no-resale equilibrium conditions (3), the two equilibrium cutoffs are

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<sup>20</sup>In the asymmetric no-resale equilibrium we construct, if some bidders never participate in the auction regardless of their values and at least two other bidders participate with positive probability (i.e., if  $b^* = 1$  and  $l > 1$ ), revenue would be higher in the corresponding equilibrium with resale (since the participating bidders will enter with higher probability and bid more). On the other hand, allowing for resale would eliminate all revenue if the asymmetric equilibrium with resale has only one bidder participating with a positive probability (i.e., if  $b^{**} = 1$  and  $l = 1$ ).

identified as  $b^* \in [v_s, 1]$  and  $a^* = \Phi(b^*) \in (0, v_s]$  such that  $g(b^*) \leq 0$ , with equality if  $b^* < 1$ . This confirms the existence of the symmetric equilibrium at  $a^* = b^* = v_s$ . Tan and Yilankaya (2006) show that when  $F$  is strictly convex, function  $g(\cdot)$  is decreasing around  $v_s$ . This implies that the equilibrium conditions are satisfied for at least one pair of asymmetric cutoffs: *Either* there exists a  $b^* \in (v_s, 1)$  such that  $g(b^*) = 0$  (thus there exists an asymmetric equilibrium with  $b^*$  as the high cutoff and  $\Phi(b^*)$  as the low cutoff), *or*  $g(1) \leq 0$  (thus there exists an asymmetric equilibrium with 1 as the high cutoff and  $\Phi(1)$  as the low cutoff).

Now consider the smallest value of  $b$  that satisfies these equilibrium conditions:  $\bar{b} = \min \{b > v_s : g(b) \leq 0, \text{ with equality if } b < 1\}$ . To see that surplus is higher under cutoffs  $\bar{b}$  and  $\Phi(\bar{b})$ , write  $\tilde{S}(\Phi(\bar{b}), \bar{b}) - \tilde{S}(v_s, v_s)$  as

$$\begin{aligned} & \int_{v_s}^{\bar{b}} \left( \frac{\partial \tilde{S}(\Phi(b), b)}{\partial b} + \frac{\partial \tilde{S}(\Phi(b), b)}{\partial a} \Phi'(b) \right) db \\ &= \int_{v_s}^{\bar{b}} (-hf(b) \underbrace{[\tilde{\pi}_H(\Phi(b), b) - c]}_{g(b)} - lf(a) \underbrace{[\tilde{\pi}_L(\Phi(b), b) - c]}_0) \Phi'(b) db \\ &= \int_{v_s}^{\bar{b}} -hf(b) g(b) db \end{aligned}$$

By definition of  $\bar{b}$ , the integrand is positive for all  $b \in (v_s, \bar{b})$ , proving that  $S(\Phi(\bar{b}), \bar{b}) > S(v_s, v_s)$ . ■

**Lemma 1** *Consider a low-cutoff bidder with value  $w \in [a, b]$  who won the initial auction. Her expected payoff from the resale phase is*

$$\beta(w) = b - \int_w^b \frac{F(r(x))^h}{F(b)^h} dx. \quad (18)$$

**Proof** Consider the standard optimal auction problem where the seller's value is  $w \in [a, b]$ , there are  $h \geq 1$  bidders whose valuations are independently distributed on  $[0, b]$  according to cdf  $\frac{F(\cdot)}{F(b)}$ , where  $0 \leq a \leq b \leq 1$ . Note that (18) is just the expected payoff of the seller, obtained from the standard formulation (see Myerson (1981)) by using a change of variables to incorporate the reserve price. Here we give a heuristic argument as well: The revenue equivalence theorem implies that the continuation payoff for the winner of the initial auction is a continuous function of her valuation and its derivative at each valuation is equal to the probability that the bidder will keep the good at

the end of the resale phase. Bidders with  $v \geq b$  will not resell the object if they win the initial auction. Accordingly, the continuation payoff of a bidder with valuation  $b$  is equal to  $b$ . A reseller with value  $w \in [a, b]$ , who sets her reserve price optimally to  $r(w)$ , does not sell the object if all  $h$  bidders have valuations less than  $r(w)$ , which happens with probability  $\frac{F(r(w))^h}{F(b)^h}$ . Accordingly,

her continuation payoff is  $b - \int_w^b \frac{F(r(x))^h}{F(b)^h} dx$ . ■

**Lemma 2** *Suppose that all bidders except one are following their equilibrium strategies. For the remaining bidder, the payoff differential between participation in the initial auction and staying out of it is weakly increasing in her valuation.*

**Proof** It follows from the revenue equivalence theorem that a bidder’s “non-participation” payoff is continuous in her valuation and its derivative is equal to the probability that this bidder will acquire the auctioned object during the resale phase. Her “participation payoff” is continuous in her valuation as well and its derivative is equal to the probability that she receives the good and keeps it after the end of the initial auction and the resale phase. To conclude that the payoff differential is weakly increasing in valuation, it will be sufficient to show that probability of acquiring the good is at least as large for all bidder valuations if the bidder were to participate in the initial auction.

Suppose that the bidder’s valuation is  $v$ . If this bidder stays out of the initial auction, she will acquire the object only when the valuations of all the high-cutoff bidders (excluding the bidder in question, in case that she is a high-cutoff bidder) are lower than  $v$  and the valuation of the highest low-cutoff bidder (excluding the bidder in question) is between  $a$  and  $r^{-1}(v)$ . Now notice that, when the valuations of the other bidders satisfy this condition, the same bidder would have acquired the object by participating in the initial auction (either by overbidding the highest low-cutoff bidder or at the resale phase) as well. This proves that entering in the initial auction does not decrease the probability of acquiring the object for any valuation. ■

**Proof of Remark 2**

*i)  $\pi_L(a, b)$  is increasing in  $a$  and  $b$  for  $a > 0$ .*

Using (9) and (11), we have

$$\pi_L(a, b) = F(a)^{l-1} F(b)^h \beta(a) = F(a)^{l-1} F(b)^h \left[ b - \int_a^b \frac{F(r(x))^h}{F(b)^h} dx \right].$$

Let  $a > 0$ .  $\pi_L(a, b)$  is increasing in  $a$ , since  $\beta(a)$  is increasing in  $a$ .

$$\begin{aligned}
\frac{\partial \pi_L(a, b)}{\partial b} &= hF(a)^{l-1} [F(b)^{h-1} f(b) b - \int_a^b F(r(x))^{h-1} f(r(x)) \frac{\partial r(x)}{\partial b} dx] \\
&= hF(a)^{l-1} f(b) [F(b)^{h-1} b - \int_a^b F(r(x))^{h-1} r'(x) dx] \\
&\geq hF(a)^{l-1} f(b) [F(b)^{h-1} b - \int_a^b F(b)^{h-1} dx] \\
&= hF(a)^{l-1} f(b) F(b)^{h-1} a > 0,
\end{aligned}$$

where the second equality follows from  $f(r(x)) \frac{\partial r(x)}{\partial b} = f(b) r'(x)$  (using the implicit function theorem for (7)), and the inequality follows from  $F(b) \geq F(r(x))$  and  $r'(x) \in (0, 1)$ .

ii)  $\pi_H(a, b)$  is increasing in  $a$ .

From (14),

$$\frac{\partial \pi_H(a, b)}{\partial a} = lF(a)^{l-1} f(a) [F(b)^{h-1} b - \alpha(b, r(a)) + \beta(a)].$$

Using (8), (9), and a change of variables ( $x \rightarrow r(x)$ ),

$$\begin{aligned}
\frac{\partial \pi_H(a, b)}{\partial a} &= lF(a)^{l-1} f(a) [F(b)^{h-1} b - \frac{1}{F(b)} \int_a^b F(r(x))^{h-1} (F(r(x)) - F(b) r'(x)) dx] \\
&\geq lF(a)^{l-1} f(a) [F(b)^{h-1} b - \int_a^b F(b)^{h-1} dx] \\
&= lF(a)^{l-1} f(a) F(b)^{h-1} a > 0,
\end{aligned}$$

where the inequality follows  $F(b) \geq F(r(x))$  and  $r'(x) \in (0, 1)$ . ■

### **Proof of Proposition 2 (Existence of Asymmetric Equilibria with Resale)**

The proof will use the following lemma.

**Lemma 3** *If  $0 < a < b$  then  $\pi_L(a, b) > \tilde{\pi}_L(a, b)$  and  $\pi_H(a, b) < \tilde{\pi}_H(a, b)$ .*

**Proof** Recall that ((1),(2),(11),(14))

$$\begin{aligned}
\tilde{\pi}_L(a, b) &= F(a)^{l-1} F(b)^h a \\
\pi_L(a, b) &= F(a)^{l-1} F(b)^h \beta(a) \\
\tilde{\pi}_H(a, b) &= F(b)^{h-1} [F(a)^l b + \int_a^b (b-w) dF(w)^l] \\
\pi_H(a, b) &= F(b)^{h-1} [F(a)^l b + \int_a^b [\alpha(b, r(w)) - \beta(w)] dF(w)^l].
\end{aligned}$$

Let  $0 < a < b$ . The first inequality follows from  $\beta(a) > a$ . The second inequality follows from  $b \geq \alpha(b, r(w))$  and  $\beta(w) > w$  for  $w \in [a, b]$ . ■

**Proof of Proposition 2.** For all  $b \in [v_s, 1]$ , define

$$\Phi(b) = \begin{cases} a : \pi_L(a, b) = c & \text{if } \pi_L(0, b) < c \\ 0 & \text{otherwise} \end{cases}$$

Notice that  $\Phi(b)$  is continuously differentiable, strictly decreasing whenever it takes positive values, and that  $\Phi(v_s) = v_s$ . For all  $b \in [v_s, 1]$ , also define

$$g(b) = \pi_H(\Phi(b), b) - c.$$

This last function is also continuously differentiable with  $g(v_s) = 0$ . The resale equilibrium conditions are satisfied (with  $b$  as the high cutoff and  $\Phi(b)$  as the low cutoff) if and only if  $g(b) \leq 0$ , with equality if  $b < 1$ .

Recall that  $\tilde{\pi}_L(a^*, b^*) = c$ . Since function  $\pi_L(a, b)$  is increasing in  $a$  and is larger than  $\tilde{\pi}_L(a, b)$  for  $0 < a < b$ , it must be that  $\Phi(b^*) < a^*$ . Consider

$$g(b^*) = \pi_H(\Phi(b^*), b^*) - c < \pi_H(a^*, b^*) - c < \tilde{\pi}_H(a^*, b^*) - c \leq 0$$

The first inequality follows from monotonicity of  $\pi_H$  in its first argument, and the second one from  $\pi_H(a, b) < \tilde{\pi}_H(a, b)$  for  $0 < a < b$ . Finally,  $g(b^*) < 0$  implies the existence of  $b^{**} \geq b^*$  (with strict inequality if  $b^* < 1$ ) such that  $g(b^{**}) \leq 0$  (with equality if  $b^{**} < 1$ ). Accordingly, there exists an equilibrium where  $h$  bidders use cutoff  $b^{**}$  and  $l$  bidders use cutoff  $a^{**} = \Phi(b^{**})$ . To complete the proof, notice that  $a^{**} \leq \Phi(b^*) < a^*$ . ■

**Proof of Proposition 3 (Welfare under Resale)**

The proof of the proposition will follow from these lemmas:

**Lemma 4**  $\frac{\partial S(a,b)}{\partial a} \leq -lf(a) [\pi_L(a, b) - c]$ .

**Proof**

$$\begin{aligned}\frac{\partial S(a, b)}{\partial a} &= -[F(r(a))^h a + \int_{r(a)}^b v dF(v)^h] l F(a)^{l-1} f(a) + l f(a) c \\ &= -l f(a) [F(r(a))^h F(a)^{l-1} a + \int_{r(a)}^b F(a)^{l-1} v dF(v)^h - c]\end{aligned}$$

Recalling that  $\pi_L(a, b) = F(a)^{l-1} F(b)^h \beta(a) = F(a)^{l-1} F(b)^h [b - \int_w^b \frac{F(r(x))^h}{F(b)^h} dx]$ , we need to show

$$\begin{aligned}F(a)^{l-1} [F(b)^h b - F(r(a))^h a - \int_a^b F(r(x))^h dx] &\leq F(a)^{l-1} \int_{r(a)}^b v dF(v)^h, \text{ or} \\ F(a)^{l-1} \int_a^b x dF(r(x))^h &\leq F(a)^{l-1} \int_a^b r(x) dF(r(x))^h,\end{aligned}$$

where the left hand side is obtained by using integration by parts and the right hand side a change of variables. The inequality holds since  $x < r(x)$  for all  $x < b$ . ■

**Lemma 5** *If  $\frac{vf(v)}{F(v)}$  is weakly increasing in  $v$ , then  $\frac{\partial S(a, b)}{\partial b} \leq -hf(b) [\pi_H(a, b) - c]$ .*

**Proof**

- **Step 1:** If  $\frac{vf(v)}{F(v)}$  is weakly increasing in  $v$ , then  $F(r(w)) - F(b) r'(w) \geq 0$  for all  $w < b$ , and the inequality is strict if  $w > b$ .

Recall the formula of the reserve price

$$r(w) = w + \frac{F(b) - F(r(w))}{f(r(w))}$$

Total differentiation reveals  $r'(w)$  as

$$r'(w) = \frac{1}{2 + [F(b) - F(r)] \frac{f'(r)}{f(r)^2}}$$

where  $r$  equals the optimal reserve price for valuation  $w$ . So we need to show

$$\begin{aligned}F(b) &\leq \frac{F(r(w))}{r'(w)} = 2F(r) + [F(b) - F(r)] \frac{f'(r)}{f(r)^2} F(r) \\ F(b) - F(r) &\leq F(r) + [F(b) - F(r)] \frac{f'(r)}{f(r)^2} F(r)\end{aligned}$$



which is identical to

$$\begin{aligned} \left[1 - \frac{f'(r) F(r)}{f(r)^2}\right] [F(b) - F(r)] &\leq F(r) \\ \left[\frac{f(r)^2 - f'(r) F(r)}{f(r)}\right] [r - w] &\leq F(r) \end{aligned}$$

The last inequality follows from the formula for the reserve price. Recall that  $r - w \geq 0$ . If  $f(r)^2 \leq f'(r) F(r)$ , then the left hand side is at most zero and the inequality is satisfied. Otherwise, showing the inequality below is sufficient for the proof.

$$\left[\frac{f(r)^2 - f'(r) F(r)}{f(r)}\right] r \leq F(r)$$

or  $F(r) f(r) + f'(r) F(r) r - f(r)^2 r \geq 0$ . The left hand side is equal to the numerator of the derivative of  $\frac{r f(r)}{F(r)}$  with respect to  $r$ . The denominator of the derivative is  $F(r)^2 > 0$ . Therefore if  $\frac{v f(v)}{F(v)}$  is weakly increasing in  $v$  for all values lower than  $b$ , then  $F(r(w)) - F(b) r'(w) \geq 0$ , concluding the proof of this first step.

- **Step 2:** If  $F(r(w)) - F(b) r'(w) \geq 0$  for all  $w \in [0, b]$ , then  $\frac{\partial S(a,b)}{\partial b} \leq -h f(b) [\pi_H(a, b) - c]$ .

$$\begin{aligned} \frac{\partial S(a, b)}{\partial b} &= -hbF(b)^{h+l-1} f(b) + hf(b)c \\ &\quad + \int_a^b \left[ \begin{array}{c} hbF(b)^{h-1} f(b) + hwF(r(w))^{h-1} f(r(w)) \frac{dr(w)}{db} \\ -hr(w) F(r(w))^{h-1} f(r(w)) \frac{dr(w)}{db} \end{array} \right] dF(w)^l \end{aligned}$$

Recall that

$$\begin{aligned} \pi_H(a, b) &= F(a)^l F(b)^{h-1} b + F(b)^{h-1} \int_a^b [\alpha(b, r(w)) - \beta(w)] dF(w)^l \\ &= F(a)^l F(b)^{h-1} b + \frac{1}{F(b)} \int_a^b \int_w^b F(r(x))^{h-1} [F(r(x)) - F(b) r'(x)] dx dF(w)^l \end{aligned}$$

Now we have to show

$$\begin{aligned}
& F(a)^l F(b)^{h-1} b + \frac{1}{F(b)} \int_a^b \int_w^b F(r(x))^{h-1} [F(r(x)) - F(b) r'(x)] dx dF(w)^l \\
& \geq F(b)^{h-1} F(a)^l b + \int_a^b (r(w) - w) F(r(w))^{h-1} r'(w) dF(w)^l
\end{aligned}$$

The first terms on either side of the inequality cancel out. Since  $[F(r(x)) - F(b) r'(x)]$  is non-negative for  $x \leq b$  and  $F(r(x))^{h-1} \geq F(r(w))^{h-1}$  for  $x \geq w$ , it is sufficient to show

$$\begin{aligned}
& \frac{1}{F(b)} \int_a^b F(r(w))^{h-1} \int_w^b [F(r(x)) - F(b) r'(x)] dx dF(w)^l \\
& \geq \int_a^b (r(w) - w) F(r(w))^{h-1} r'(w) dF(w)^l
\end{aligned}$$

$F(r(x)) - F(b) r'(x)$  can be written as

$$[F(r(x)) - F(b)] r'(x) + F(x) [1 - r'(x)] = \frac{[F(r(x)) - F(b)]}{f(r(x))} f(r(x)) r'(x) + F(r(x)) [1 - r'(x)]$$

Using the formula  $\frac{[F(r(x)) - F(b)]}{f(r(x))} = x - r(x)$

$$\begin{aligned}
& = [x - r(x)] f(r(x)) r'(x) + F(r(x)) [1 - r'(x)] \\
& = \frac{d[x - r(x)] F(r(x))}{dx}
\end{aligned}$$

Therefore  $\int_w^b [F(r(x)) - F(b) r'(x)] = [r(w) - w] F(r(w))$ . So showing the below inequality would be sufficient for the result:

$$\frac{1}{F(b)} \int_a^b F(r(w))^h (r(w) - w) dF(w)^l \geq \int_a^b (r(w) - w) F(r(w))^{h-1} r'(w) dF(w)^l,$$

which can be rewritten as

$$\int_a^b [F(r(w)) - F(b) r'(w)] F(r(w))^{h-1} (r(w) - w) dF(w)^l \geq 0$$

This inequality holds since all terms in the integrand are positive for all  $w \in (a, b)$  under the hypothesis of the lemma. ■

**Proof of Proposition 3.** We know from the proof of the Proposition 2 that  $g(b^*) < 0$  and there exists  $b$  which satisfies the resale equilibrium conditions with  $\Phi(b)$ . Now consider the smallest such value of for  $b$ :  $b^{**} = \min \{b \geq b^* : g(b) \leq 0, \text{ with equality if } b < 1\}$ . To see that surplus is higher under cutoffs  $b^{**}$  and  $\Phi(b^{**})$ , write  $S(\Phi(b^{**}), b^{**}) - S(a^*, b^*)$  as

$$\begin{aligned} & \int_{a^*}^{\Phi(b^{**})} \frac{\partial S(a, b^*)}{\partial a} da + \int_{b^*}^{b^{**}} \left( \frac{\partial S(\Phi(b), b)}{\partial b} + \frac{\partial S(\Phi(b), b)}{\partial a} \Phi'(b) \right) db \\ \geq & \int_{\Phi(b^*)}^{a^*} lf(a) [\pi_L(a, b^*) - c] da + \int_{b^*}^{b^{**}} -hf(b) [\pi_H(\Phi(b), b) - c] db \\ & + \int_{b^*}^{b^{**}} -lf(a) [\pi_L(\Phi(b), b) - c] \Phi'(b) db \end{aligned}$$

The inequality above follows from the previous two lemmas and that  $\Phi'(b) \leq 0$ . Moreover,  $\pi_L(a, b^*) > \pi_L(\Phi(b^*), b^*) = c$  for all  $a \in (\Phi(b^*), a^*]$ , establishing that the first integral above is strictly positive. The last integral is non-negative since  $\pi_L(\Phi(b), b) \geq c$ . Finally, the term  $[\pi_H(\Phi(b), b) - c]$  in the second integral equals to  $g(b)$ , which takes non-positive values for  $b \in [b^*, b^{**}]$  by definition of  $b^{**}$ . ■

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