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A Concave Security Market Line

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Abstract

We provide theoretical and empirical arguments in favor of a concave shape for the security market line, or a diminishing marginal premium for market risk. In capital market equilibrium with binding portfolio restrictions, different investors generally hold different sets of risky securities. Despite the differences in composition, the optimal portfolios generally share a joint exposure to systematic risk. Equilibrium in this case can be approximated by a concave relation between expected return and market beta rather than the traditional linear relation. An empirical analysis of U.S. stock market data confirms the existence of a significant and robust, concave cross-sectional relation between average return and estimated past market beta. We estimate that the market-risk premium is at least five to six percent per annum for the average stock, substantially higher than conventional estimates.
Keywords: capital market equilibrium, asset pricing, investment restrictions, portfolio theory.
JEL Classification: G12, C21.

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Introduction

The Capital Asset Pricing Model (Sharpe (1964), Lintner (1965), Mossin (1966) and Treynor (1961)) predicts a linear relation between the expected return and the market beta of securities - the Security Market Line. This linear relation also arises as a special case in Ross' (1976) Arbitrage Pricing Theory if one assumes that security returns obey a single-factor market model and arbitrage opportunities do not exist. A wealth of empirical research suggests that linearity should be rejected. For example, Fama and French (1992) convincingly show that small-cap stocks carry a return premium that is unrelated to market beta and that the empirical return-risk relation is flat after controlling for market capitalization. The debate about the correct interpretation of these findings is ongoing. Are historical return data and common market indexes representative for the real thing? Is or was the stock market temporarily out of equilibrium? Does market equilibrium require linearity to begin with? These and other fundamental questions will probably continue to determine the research agenda going forward.

The CAPM assumes a perfect capital market without restrictions on, for example, borrowing and short selling. Borrowing at the riskless rate to finance a risky stock portfolio generally is not realistic in a world where even secured loans carry interest rates above Treasury yields and buying stocks on margin involves even higher rates and additional restrictions. Short selling involves costs and collateral requirements that seem restrictive for many stocks and investors. Industry regulation and client mandates limit the use of leverage and short selling by most institutional investors and mutual funds. Modern-day hedge funds are empowered to use leverage and short-selling, but also their reach is limited. In addition, hedge funds represent only a fraction of the total invested wealth and they played no significant role during most of the sample periods covered by our data sets.

It is well documented that the CAPM generally breaks down in case of restrictions.

Black (1972) analyzes market equilibrium with restrictions on borrowing. In this case, the SML remains linear, although the intercept increases and the slope decreases compared with the CAPM predictions. Linearity generally breaks down if investors also face restrictions on the risky securities, such as short selling constraints. Ross (1977) and Sharpe (1991) stressed this important point before, but they stopped short of deriving an alternative shape for the SML. This study extends their analysis by providing theoretical and empirical arguments in favor of a concave shape for the SML, or a diminishing marginal premium for market risk.

Linearity is a necessary and sufficient condition for mean-variance efficiency of the market portfolio. However, deviations from linearity are not always good measures of deviations from efficiency, and vice versa. It follows from Roll (1977), Roll and Ross (1994) and Kandel and Stambaugh (1995) that a relatively small (large) reduction in expected return or increase in standard deviation of the market index can sometimes lead to relatively large (small) deviations from the classical SML. This study focuses on the cross-sectional mean-beta relation rather than evaluating market portfolio efficiency. In the case without portfolio restrictions, portfolio efficiency can be measured in an economically meaningful way by means of the Sharpe ratio. However, in the case with restrictions, the distance from the efficient frontier generally is not a meaningful degree measure for deviations from optimizing behavior. In addition, measuring the distance from the frontier requires an explicit specification of the relevant set of investment restrictions, whereas we eye results that apply more generally for every set of restrictions. We therefore do not attempt to measure deviations from market portfolio efficiency, but focus on the general shape of the return-risk relation.

Our theoretical analysis in Section I develops an extension of the CAPM with investment restrictions in the spirit of Sharpe (1991). In our model, different investors generally include different sets of risky securities in their portfolios (and thus deviate from the market portfolio). Despite the differences in their composition, the individual portfolios generally will be positively correlated and share a common exposure to systematic risk. The general equilibrium conditions appear difficult to test empirically without detailed information about the composition of the individual portfolios and the distribution of risk tolerance and wealth, a common problem for heterogeneous-investor models. However, due to the investors' common exposure to systematic risk, equilibrium can be approximated by an increasing and concave relation between expected return and the traditional market beta - a concave security market line (CSML). This relation is a generalization of the general linear SML derived by Black (1972). The concave SML becomes linear if no investor faces binding restrictions for the risky securities, and the CAPM arises if riskless borrowing is also allowed.

Our empirical analysis in Section II applies the two-pass regression methodology of Fama and MacBeth (1973) to U.S. stock market data. Following the original study, we use the squared market beta as a natural measure of SML concavity; similar results are obtained for alternative concavity measures. Whereas we find no significant linear cross-sectional relation between average returns and past beta estimates, we do find a significant and robust concave relation, consistent with our hypothesis. The inclusion of beta-squared in the regression yields an estimated market-risk premium of at least five to six percent per annum for the average stock, substantially higher than conventional estimates. In addition, beta-squared has a significantly negative coefficient, implying that the risk premium increases at a diminishing rate. Encouragingly, the concave pattern is robust to the inclusion of other stock characteristics and the selection of the cross-section and sample period. Concavity arises both in the analysis of individual stocks and for aggregated stock portfolios with stable size and beta properties, reducing possible concerns about estimation error and time-variation of stock-level betas.

I Theory

A Preliminaries

We will first introduce and motivate our assumptions. We build on the earlier work of Sharpe (1991). It is not our objective to develop the most general asset pricing model, but rather to explore the effect of binding investment restrictions while maintaining a set of simplifying assumptions that support a linear SML in the absence of binding restrictions.

Assumption 1 (Securities)**.** *The investment universe consists of $N+1$ base assets, associated with returns $\mathbf{x} \in \mathbb{R}^{N+1}$. Throughout the text, we will use the index set $\mathcal{I} = \{1, \ldots, N+1\}$ to denote the different securities. The returns have mean $\mathbb{E}[\mathbf{x}] = \boldsymbol{\mu}$ and variance-covariance matrix $\mathbb{E}\left[(\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})^T\right] = \boldsymbol{\Omega}$. Security $N+1$ is risk-free and yields a sure return of r_F.*

Assumption 2 (Investors)**.** *There are K investors. Investor k's wealth expressed as a proportion of the total wealth of all investors is $w_k > 0$. Investors may diversify between the securities, and we will use $\boldsymbol{\lambda}_k \in \mathbb{R}^{N+1}$ for the vector of optimal portfolio weights of investor k. We will use the index set $\mathcal{K} = \{1, \ldots, K\}$ to denote all investors. Investors possess mean-variance preferences, that is, the expected utility of wealth associated with a portfolio $\boldsymbol{\lambda} \in \mathbb{R}^{N+1}$ is*

$$U_{\zeta_k}(\boldsymbol{\lambda}) = \boldsymbol{\mu}^T \boldsymbol{\lambda} - \frac{1}{2\,\zeta_k} \boldsymbol{\lambda}^T \boldsymbol{\Omega} \boldsymbol{\lambda},$$

where $\zeta_k > 0$ is investor k's risk tolerance. The limiting cases with $\zeta_k \to \infty$ and $\zeta_k \to 0$ are risk neutrality and extreme risk aversion, respectively. We will use $\mu_{\boldsymbol{\lambda}}$ and $\sigma^2_{\boldsymbol{\lambda}}$ to denote the expected return and variance, respectively, of portfolio $\boldsymbol{\lambda} \in \mathbb{R}^{N+1}$.

Mean-variance preferences can equivalently be represented by a quadratic utility function. Levy and Markowitz (1979) convincingly show that this specification generally gives an accurate second-order Taylor series approximation for any well-behaved utility function on the typical return interval for stock portfolios. In some cases, the more general criteria of stochastic dominance seem more appropriate than the mean-variance rule. In this more general case, our arguments will be even more relevant, because differences in the general shape of the investors' utility functions will represent an additional source of non-convexity and market portfolio inefficiency.

Assumption 3 (Portfolios)**.** *The portfolio possibilities are represented by the simplex*

$$\Lambda = \{\boldsymbol{\lambda} \in \mathbb{R}^{N+1} : \boldsymbol{\lambda} \geq \mathbf{0}_{N+1}, \mathbf{1}_{N+1}^T \boldsymbol{\lambda} = 1\},$$

where $\mathbf{0}_{N+1}$ *and* $\mathbf{1}_{N+1}$ *are the zero and unity vectors, respectively, with dimension* $N+1$*. We use* $\mathcal{A}_{\boldsymbol{\lambda}} = \{i : \lambda_i > 0, i \in \mathcal{I}\}$ *for the "active set" of portfolio* $\boldsymbol{\lambda}$*, or all securities that are included in the portfolio with a strictly positive weight.*

The simplex Λ is the convex hull of the base assets and covers the important special case without short selling and borrowing, a natural starting point for analyzing the effects of investment restrictions. It is straightforward to generalize our analysis to the case where the portfolio possibilities are a general polytope. The Minkowski-Weyl Theorem says that any polytope can be represented as the convex hull of its vertices. Therefore, we can generalize our model by simply replacing the set of base assets with the set of the vertices, or extreme portfolios. This approach would allow us to include additional restrictions, such as position limits or restrictions on risk-factor loadings, or to relax restrictions, for instance, allowing bounded short-sales or bounded borrowing. The effect of the set of restrictions works through the set of binding restrictions in the investors' portfolios. Since the precise composition of the individual portfolios is not specified, the precise specification of the set of restrictions is inconsequential in our analysis.

For the sake of simplicity, we assume that all investors face the same set of investment restrictions. In the more general case with heterogenous restrictions, our arguments appear even more relevant, as differences in the efficient sets between investors would represent an additional source of non-convexity and market portfolio inefficiency. Although we keep the portfolio possibilities constant, the set of restrictions that is binding depends on the level of risk tolerance and will differ from one investor to another.

Assumption 4 (Market portfolio)**.** *The markets for the risky securities clear, so that all risky securities are held by the K investors in the economy. We will use $\boldsymbol{\tau} \in \mathbb{R}^{N+1}$ for the vector of portfolio weights of the market portfolio, where $\tau_i = \sum_{k \in \mathcal{K}} w_k \, \lambda_{i,k}/(1 - \lambda_{N+1,k})$ for $i = 1, \ldots, N$ and $\tau_{N+1} = 0$. The market portfolio is feasible, or $\boldsymbol{\tau} \in \Lambda$, and all risky securities have a strictly positive market capitalization, or $\tau_i > 0$ for all $i = 1, \ldots, N$.*

Assumption 5 (Return generating process)**.** *The returns of the risky securities obey the following general risk factor model*

$$x_i = a_i + \sum_{l=1}^{L} b_{i,l} \, f_l + \epsilon_i \quad \forall i = 1, \ldots N, \tag{1}$$

where f_l, $l = 1, \ldots, L$ are systematic risk factors with $\mathbb{E}\left[(f_l - \mathbb{E}\left[f_l\right])^2\right] = \sigma^2_{f_l}$, $b_{i,l}$ are the factor loadings of security i, $a_i = \mu_i - \sum_{l=1}^{L} b_{i,l} \, \mathbb{E}\left[f_l\right]$, ϵ_i is an idiosyncratic random factor for security i, with $\mathbb{E}\left[\epsilon_i\right] = 0$, $\mathbb{E}\left[f_l \, \epsilon_i\right] = 0$ for all l and $\mathbb{E}\left[\epsilon_i \, \epsilon_{i'}\right] = 0$ for all $i \neq i'$. Without loss of generality, we assume that the factors are orthogonal, or $\mathbb{E}\left[f_l \, f_p\right] = 0$, and the market return is the first factor, or $f_1 = \mathbf{x}^T \boldsymbol{\tau}$. We use $\mathbf{b}_l = (b_{1,l}, \ldots, b_{N,l})^T$ for the vector of factor loadings, $\rho_{i,l}$ for the correlation between security i and factor l, $R^2_i = \sum_{l=1}^{L} \rho^2_{i,l}$ for the percentage explained variance, and $r_{i,l} = \rho_{i,l}/R_i$ for a standardized correlation coefficient that yields $\sum_{l=1}^{L} r^2_{i,l} = 1$.

A common specification is to use the market return as the single risk factor ($L = 1$), or the "market model". This approach is particularly relevant in the context of analyzing and testing efficiency and linearity. For example, the classical Gibbons, Ross, and Shanken (1989) test for market portfolio efficiency without portfolio restrictions assumes the market model. The market model is also of interest because it implies the classical linear SML as a special case in Ross' (1976) Arbitrage Pricing Theory (APT) if we assume away investment restrictions and exclude arbitrage possibilities. Empirically, it appears that the market return explains the bulk of the joint variation of stock returns, and additional risk factors, such as Fama and French' (1993) hedge portfolio returns, explain much smaller (but sometimes significant) amounts. This finding is not surprising given that the market portfolio by construction is extremely diversified and therefore will be highly correlated with the first principal component of security returns, irrespective of the nature of the underlying risk factor model. Brown (1989) shows that we cannot always rely on statistical methods to correctly identify the relevant number of factors based on ex-post security returns. However, using ex-ante analysis, MacKinlay (1995) shows that omitted risk factors are unlikely to cause detectable deviations from the CAPM. Despite the arguments for the market model, our analysis allows securities to have a joint exposure to non-market factors.

B General Security Market Line (GSML)

The above assumptions allow us to define portfolio optimality and mean-variance efficiency and to derive and analyze equilibrium conditions. A given portfolio $\boldsymbol{\lambda} \in \Lambda$ is optimal for an investor with risk tolerance $\zeta > 0$ if and only if it maximizes expected utility:

$$\boldsymbol{\lambda} = \arg\max_{\boldsymbol{\kappa} \in \Lambda} U_{\zeta}(\boldsymbol{\kappa}). \tag{2}$$

Following Sharpe (1991, Equation 12), we will make use of the Karush-Kuhn-Tucker (KKT) conditions for this optimization problem:

$$\mu_i = \frac{1}{\zeta} \operatorname{Cov}\left(x_i, \mathbf{x}^T \boldsymbol{\lambda}\right) + \theta_{\boldsymbol{\lambda}} - \alpha_{i,\boldsymbol{\lambda}}, \quad i \in \mathcal{I} \tag{3}$$

$$\alpha_{i,\boldsymbol{\lambda}} \geq 0, \quad \forall i \in \mathcal{I} \tag{4}$$

$$\alpha_{i,\boldsymbol{\lambda}} \lambda_i = 0, \quad \forall i \in \mathcal{I} \tag{5}$$

$$\mathbf{1}_N^T \boldsymbol{\lambda} = 1, \quad \lambda_i \geq 0, \quad \forall i \in \mathcal{I}. \tag{6}$$

In these conditions, $\theta_{\boldsymbol{\lambda}} \in \mathbb{R}$ and $\boldsymbol{\alpha}_{\boldsymbol{\lambda}} = (\alpha_{1,\boldsymbol{\lambda}}, \ldots, \alpha_{N+1,\boldsymbol{\lambda}})^T \in \mathbb{R}^{N+1}$ are Lagrange multipliers that measure the shadow prices of the budget constraint and no-short-selling/no-borrowing constraints, respectively. The complementary slackness condition (5) implies that the active securities ($i \in \mathcal{A}_{\boldsymbol{\lambda}}$) must have a zero shadow price for the associated short-sales restriction ($\alpha_{i,\boldsymbol{\lambda}} = 0$), whereas strictly positive shadow prices are allowed for inactive securities ($\alpha_{i,\boldsymbol{\lambda}} \geq 0$ for $i \notin \mathcal{A}_{\boldsymbol{\lambda}}$). Each of the active securities must therefore have the same marginal utility. If this were not the case, it would be possible to increase expected utility without violating the investment restrictions by shifting wealth from an active security with a relatively low marginal utility to an active security with a relatively high marginal utility. The common value of marginal utility for the active securities is the shadow price $\theta_{\boldsymbol{\lambda}}$, or the investor's marginal utility of wealth. The inactive securities must have a marginal utility less than or equal to of the active securities, and $\boldsymbol{\alpha}_{\boldsymbol{\lambda}}$ measures their shortfall. If this were not the case, it would be possible to increase expected utility without violating any restrictions by shifting wealth from an active security to an inactive security. The availability of the riskless security implies $r_F = \theta_{\boldsymbol{\lambda}} - \alpha_{N+1,\boldsymbol{\lambda}}$, and therefore, the shadow price of the budget constraint cannot fall below the risk-free rate: $\theta_{\boldsymbol{\lambda}} \geq r_F$. If the borrowing restriction is not binding for portfolio $\boldsymbol{\lambda}$, the complementary slackness condition (5) implies that $\alpha_{N+1,\boldsymbol{\lambda}} = 0$ and thus $\theta_{\boldsymbol{\lambda}} = r_F$.

For a risky portfolio $\boldsymbol{\lambda}$, that is, $\sigma^2_{\boldsymbol{\lambda}} > 0$, the optimality condition (3) can be expressed in terms of the exposure coefficient, or "beta", of security i with respect to portfolio $\boldsymbol{\lambda}$:

$$\mu_i = \left(\frac{1}{\zeta}\sigma^2_{\boldsymbol{\lambda}}\right)\beta_{i,\boldsymbol{\lambda}} + \theta_{\boldsymbol{\lambda}} - \alpha_{i,\boldsymbol{\lambda}}, \tag{7}$$

where

$$\beta_{i,\boldsymbol{\lambda}} = \frac{\mathrm{Cov}\left(x_i, \mathbf{x}^T\boldsymbol{\lambda}\right)}{\sigma^2_{\boldsymbol{\lambda}}}. \tag{8}$$

The optimality conditions (3) and (5) imply the following relation between an optimal portfolio's expected return $\mu_{\boldsymbol{\lambda}}$, its standard deviation $\sigma_{\boldsymbol{\lambda}}$, the shadow price of the budget constraint $\theta_{\boldsymbol{\lambda}}$ and the risk tolerance parameter ζ:

$$\frac{1}{\zeta}\sigma_{\boldsymbol{\lambda}}^2 = \mu_{\boldsymbol{\lambda}} - \theta_{\boldsymbol{\lambda}}. \tag{9}$$

If the borrowing restriction is not binding for portfolio $\boldsymbol{\lambda}$, the right-hand side of Equation (9) is the risk premium $\mu_{\boldsymbol{\lambda}} - r_F$ of portfolio $\boldsymbol{\lambda}$.

For cases in which investors' risk tolerance and/or optimal portfolios are not specified, it is useful to consider the mean-variance efficient set, or all portfolios that are optimal for at least some risk tolerance level:

$$\mathcal{E} = \{\boldsymbol{\lambda} \in \Lambda \,:\, \boldsymbol{\lambda} = \arg\max_{\boldsymbol{\kappa}\in\Lambda} U_\zeta(\boldsymbol{\kappa}),\, \zeta > 0\}. \tag{10}$$

Under the above assumptions, the efficient set generally is not convex. Combining multiple efficient portfolios with different sets of binding restrictions generally produces an inefficient combined portfolio. The efficient set contains convex neighborhoods of efficient portfolios with the same sets of binding restrictions, but combining elements from these efficient subsets generally yields inefficient portfolios. As a case in point, in the numerical example in Section ID, a portfolio of low-beta stocks is optimal for conservative investors and a portfolio of high-beta stock is optimal for adventurous investors; however, diversification across the two efficient portfolios lead to an inefficient portfolio that is not optimal for moderate investors.

The aggregate market portfolio is a weighted average of the portfolios of all individual investors. Without convexity of the efficient set, optimizing behavior by individual investors does not guarantee that the market portfolio is optimal for a representative investor. To the contrary, given that different investors generally hold different portfolios with different sets of binding restrictions, the market portfolio generally is predicted to be inefficient. Following Sharpe (1991, Equations 13-14), we can obtain the the following general relation between securities' expected returns and their market betas by taking a wealth-weighted average of the investors' KKT conditions:

Theorem 1 (Generalized Security Market Line). *For all $i \in \mathcal{I}$*

$$\mu_i = \left(\frac{1}{\bar{\zeta}}\,\sigma^2_{\boldsymbol{\tau}}\right)\,\beta_{i,\boldsymbol{\tau}} + \bar{\theta} - \bar{\alpha}_i \tag{11}$$

where $\bar{\zeta} = \sum_{k\in\mathcal{K}} w_k\,\zeta_k/(1-\lambda_{N+1,k})$ is the societal risk tolerance, $\bar{\theta} = (1/\bar{\zeta})\,\sum_{k\in\mathcal{K}} w_k\,\zeta_k\,\theta_{\boldsymbol{\lambda}_k}/(1-\lambda_{N+1,k})$ and $\bar{\alpha}_i = (1/\bar{\zeta})\,\sum_{k\in\mathcal{K}} w_k\,\zeta_k\,\alpha_{i,\boldsymbol{\lambda}_k}/(1-\lambda_{N+1,k})$ are weighted averages of the shadow prices of the borrowing and short-sales restrictions, respectively.

The market portfolio is efficient if the short-sales constraints for the risky securities are not binding for all investors. In this case, all relevant short-sales shadow prices are zero ($\boldsymbol{\alpha}_{i,\boldsymbol{\lambda}_k} = 0$) and we obtain the classical linear relation between securities' expected returns and their market betas (SML), that is,

$$\mu_i = \left(\frac{1}{\bar{\zeta}}\,\sigma^2_{\boldsymbol{\tau}}\right)\,\beta_{i,\boldsymbol{\tau}} + \bar{\theta}. \tag{12}$$

The intercept $\bar{\theta}$ and the slope $\bar{\zeta}$ depend on the restrictions that are imposed on the risk-free security. If the borrowing restriction is not binding for any investor, then the intercept equals the risk-free rate ($\overline{\theta} = r_F$) and the slope equals the market-risk premium ($(1/\bar{\zeta})\,\sigma^2_{\boldsymbol{\tau}} = \mu_{\boldsymbol{\tau}} - r_F$; see Equation (9)). If borrowing is binding for some investors, then the intercept should be greater than or equal to the risk-free rate ($\overline{\theta} \geq r_F$) and the slope should be smaller than or equal to the market-risk premium ($(1/\bar{\zeta})\,\sigma^2_{\boldsymbol{\tau}} = \mu_{\boldsymbol{\tau}} - \overline{\theta} \leq \mu_{\boldsymbol{\tau}} - r_F$; see Equation (9)), as in Black's (1972) model.

The general non-linear GSML appears difficult to test without detailed information about individual portfolios ($\boldsymbol{\lambda}_k$) and the distribution of wealth (w_k) and risk tolerance (ζ_k), a common problem faced in heterogeneous investor models. Nevertheless, as we will show next, the GSML can be approximated by an increasing and concave function as a result of the investors' joint exposure to systematic risk.

C Concave Security Market Line (CSML)

Investors generally hold portfolios that include only a subset of the available securities and deviate from market weights. Nevertheless, the individual portfolios can be strongly correlated with the market portfolio, even if they represent only a fraction of the market, due to a common exposure to systematic risk. As a case in point, we created two monthly-rebalanced, non-overlapping stock portfolios of approximately equal market value from all U.S. common stocks included our empirical analysis (see Section II: a value-weighted portfolio of stocks with market betas below the median beta and a value-weighted portfolio of stocks with above-median betas. Although the two portfolios each represent only half of the total market, the correlation of monthly returns with the stock market index is 96 percent for the low-beta portfolio and 97 percent for the high-beta portfolio.

If the optimal portfolios have high market correlations, then the correlation of an individual security relative to a given optimal portfolio ($\rho_{i,\boldsymbol{\lambda}_k}$) will tend to be closely related to its correlation with the market portfolio ($\rho_{i,\boldsymbol{\tau}}$). To capture this pattern, we will use the following correlation ratio:

$$\xi_{i,\boldsymbol{\lambda}_k,\boldsymbol{\tau}} = \frac{\rho_{i,\boldsymbol{\lambda}_k}}{\rho_{i,\boldsymbol{\tau}}}. \tag{13}$$

It seems natural to assume that this correlation ratio shows relatively small differences between the active assets in a given optimal portfolio and that the ratio tends to be larger for active assets than for inactive assets. After all, the assets included in a given portfolio are more likely to have similar risk characteristics than the assets excluded from the portfolio. For example, a conservative investor is more likely to invest in large-cap low-beta stocks and an adventurous investor is more likely to invest in small-cap high-beta stocks. We formalize this assumption in the following way:

Assumption 6 (Correlation ratios)**.** *For all $k \in \mathcal{K}$, there exists $\xi_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}} \geq 0$ such that*

$$\xi_{i,\boldsymbol{\lambda_k},\boldsymbol{\tau}} = \xi_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}} \quad \textit{for all} \quad i \in \mathcal{A}_{\boldsymbol{\lambda}_k}, \quad \textit{and} \tag{14}$$

$$\xi_{i,\boldsymbol{\lambda_k},\boldsymbol{\tau}} \geq \xi_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}} \quad \textit{for all} \quad i \notin \mathcal{A}_{\boldsymbol{\lambda}_k}. \tag{15}$$

This correlation structure arises for several relevant special cases. It is straightforward to see that the correlation applies in the case of the CAPM with a single optimal risky portfolio ($\boldsymbol{\lambda}_k = c_k \, \boldsymbol{\tau}$). A more general sufficient condition follows:

Theorem 2 (Sufficient condition)**.** *The correlation structure (14)-(15) applies when optimal portfolios $\boldsymbol{\lambda}_k$, $k \in \mathcal{K}$, have neutral non-market factor loadings, that is, $\mathbf{b}_l^T \boldsymbol{\lambda}_k = 0$ for all $l = 2, \ldots, L$, and are well diversified, that is, $\lambda_{i,k} \, \sigma^2(\epsilon_i) \rightarrow 0$ for all $i \in \mathcal{A}_{\boldsymbol{\lambda}_k}$. Moreover, we have*

$$\xi_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}} = \frac{1}{\rho_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}}}. \tag{16}$$

The sufficient condition is satisfied in the important case that individual assets have zero non-market loadings, or the market model. As discussed before, the market model is a common specification for analyzing and testing market portfolio efficiency and linearity of the SML; see, for example, Gibbons, Ross, and Shanken (1989). However, our condition is more general and more plausible than the market model. Diversified portfolios tend to have relatively small non-market loadings compared with market betas. Risk reduction through diversification encourages the investor to combine stocks with high and low non-market loadings, moving the portfolio's non-market loadings to the average value of zero. As an extreme example, the market portfolio of all risky securities by definition has zero non-market loadings and a unity market beta. Indeed, Assumption 6 is a necessary but not sufficient condition for the CAPM. In addition, extreme loadings to known non-market risk factors (such as Fama and French' SMB and HML factors) are mostly concentrated in the micro-cap market segment and a broadly diversified portfolio generally has small non-market loadings. We stress that we do not challenge the explanatory power of non-market factors for individual stocks or for benchmark portfolios formed to have maximal exposure to non-market risk factors. We merely say that optimal portfolios are likely to have limited exposure to non-market risk. As a case in point, the market model explains 92.2 (94.5) percent of the return variation of the above-mentioned low-beta (high-beta) portfolio and the Fama-French three-factor model explains only 1.8 (0.9) percent more, and the SMB and HML loadings of these two portfolios are not significant.[1]

Under Assumption 6, we can derive an interesting special case of the GSML:

Theorem 3 (Concave Security Market Line)**.** *Under Assumption 6 the following increasing and concave relation between expected return μ_i and market beta $\beta_{i,\boldsymbol{\tau}}$ holds for all $i = 1, \ldots, N$:*

$$\mu_i = \hat{\mu}_i = \min_{k \in \mathcal{K}} \left[\left(\frac{1}{\zeta_k} \sigma_{\boldsymbol{\lambda}_k} \, \sigma_{\boldsymbol{\tau}} \, \xi_{\boldsymbol{\lambda}_k, \boldsymbol{\tau}} \right) \beta_{i,\boldsymbol{\tau}} + \theta_{\boldsymbol{\lambda}_k} \right]. \tag{17}$$

The CSML is a piecewise-linear function of market beta. Every linear line segment reflects an individual investor's portfolio optimization problem; the intercept ($\theta_{\boldsymbol{\lambda}_k}$) is the investor's shadow price of the budget constraint and the slope ($(1/\zeta_k)\, \xi_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}} \, \sigma_{\boldsymbol{\lambda}_k} \, \sigma_{\boldsymbol{\tau}}$) reflects her risk tolerance level and the covariance between her portfolio and the market portfolio. Every individual line connects the investor's active securities and supports her inactive securities from above. Minimizing across these individual, increasing and linear functions yields an overall increasing and concave, piecewise linear shape. The numerical example in Section ID shows a simple two-investor case where the SML consists of two linear line segments. In general, the number of line segments increases with the number of different investors with different active sets and the function can approximate a smooth curve in

[1]The SMB and HML factors are not orthogonal to the market return, contrary to our Assumption 5. The significance of these factors is therefore established by comparing the explanatory power of the market model and the three-factor model rather than analyzing their explanatory power in isolation.

case of a continuum of investors with different risk-tolerance levels. The empirical analysis in Section II captures the concavity, or non-linearity, of the SML with the squared market beta, first introduced in empirical asset pricing by Fama and MacBeth (1973).

The relationship is concave, because the price of a unit of risk is relatively high (low) for low-beta (high-beta) securities that enter in the portfolio of conservative (adventurous) investors but not in the portfolio of adventurous (conservative) investors. Roughly speaking, an adventurous investor can construct a portfolio of high-beta stocks to diversify away most non-systematic risk. The investor will not gain substantial additional diversification benefits from investing in low-beta stocks. In addition, adding low-beta stocks lowers the expected return to her portfolio. In a perfect capital market, she will still attempt to exploit the remaining diversification benefits, and increase the expected return by means of leverage. However, if borrowing is restricted, the prospect of lower expected returns will lower her demand for low-beta stocks and may keep her away from these stocks altogether. A conservative investor faces a similar dilemma. She can reduce non-systematic risk in an effective way by diversifying across low-beta stocks. In a perfect capital market, she will also take a position in high-beta stocks to gain extra diversification benefits. To reduce the risk associated with high-beta stocks, she would also buy Treasury bills. In our imperfect market, the adventurous investors drive down the yields of high-beta stocks relative to low-beta stocks, discouraging the conservative investor to invest in high-beta stocks.

The non-linear shape of the CSML is reminiscent of asset pricing models that assign a role to higher-order and lower-partial co-moments with the market portfolio (for example, Bawa and Lindenberg 1977). However, our model exhibits a non-linear price of market beta rather than a price for non-linear market-risk exposure, and there are important differences between these two approaches. First, the theoretical motivation is very different. The role of non-linear market-risk exposure follows from deviations from a normal return distribution and mean-variance risk preferences, whereas our non-linearity stems from differences between investors' risk tolerance and binding restrictions, an explanation that is perfectly consistent with mean-variance analysis. Second, the return distribution in typical applications seems sufficiently close to a normal distribution to apply the Levy and Markowitz (1979) argument for mean-variance analysis. Indeed, Dittmar (2002, Section IIID) and Post (2003, Section IV) show in a convincing way that higher-order and lower-partial co-moments with the market portfolio cannot rationalize market portfolio inefficiency. By contrast, the effect of binding restrictions occurs also under a normal distribution, as our model shows.

Our model allows the market portfolio to be mean-variance inefficient. In case of market portfolio inefficiency, market beta still plays a role, because the individual optimal portfolios have a strong joint exposure to market risk. As discussed in the Introduction, we do not intend to measure the economic magnitude of deviations of market portfolio efficiency. In Roll (1977), Roll and Ross (1994) and Kandel and Stambaugh (1995), (spurious) violations of market portfolio efficiency can stem from deviations of the benchmark index and the true market portfolio (benchmark error). In our model, (true) violations of efficiency arise from binding investment restrictions, as in Ross (1977) and Sharpe (1991). However, a similar caveat applies here: larger (smaller) non-linearities generally do not mean that the market portfolio is further from (closer to) the efficient frontier. In addition, a small distance from the efficient frontier generally is not "economically better" than a large distance, because the distance is not measured using to the investor's (unknown) utility function or optimal portfolio, but using the "most favorable utility function" or "closest efficient portfolio".

The CSML (17) is intended as an approximation to the complex GSML (11) that is implied by the diverse optimality conditions of investors with different levels of risk tolerance and different active sets. Our model is not intended to describe the individual optimal portfolios. In fact, if the CSML gives a perfect fit, then the portfolios of different investors are predicted to show minimal overlap. Specifically, every two linear line segments in mean-beta space can share at most one mean-beta combination. Hence, the portfolios of two different investors (with different intercepts and slopes) will overlap only for more than one security if the overlapping securities have the same market beta. Clearly, this prediction seems equally unrealistic as the CAPM prediction that all investors hold all securities and use market capitalization weights. The general GSML does allow for general overlapping portfolios and the restrictions on portfolio overlap arise from considering the limiting case of neutral optimal non-market loadings.

Deviations from the correlation structure in Assumption 6 yield the following deviations of the CSML (17) from the general GSML (11):

Theorem 4 (CSML Errors in Expected Returns)**.** *Assume that* $\xi_{i,\boldsymbol{\lambda}_k,\boldsymbol{\tau}} \geq 0$ *for all* $k \in \mathcal{K}$ *and* $i = 1, \ldots, N$. *Then*

$$|\hat{\mu}_i - \mu_i| \leq |\mu_{\boldsymbol{\lambda}^\star} - r_f| \frac{\sigma_{\boldsymbol{\tau}}}{\sigma_{\boldsymbol{\lambda}^\star}} \beta_{i,\boldsymbol{\tau}} \max_{k\,:\,i \in \mathcal{A}_{\boldsymbol{\lambda}_k}} |\xi_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}} - \xi_{i,\boldsymbol{\lambda}_k,\boldsymbol{\tau}}| \,. \tag{18}$$

where $\boldsymbol{\lambda}^\star = \arg\max_{\boldsymbol{\kappa} \in \mathcal{E}} \left(\frac{\mu_{\boldsymbol{\kappa}} - \theta_{\boldsymbol{\kappa}}}{\sigma_{\boldsymbol{\kappa}}} \right)$ *is the mean-variance tangency portfolio.*

For a given security i, the error is bounded by its correlation ratios in portfolios that include the security, or, $\xi_{i,\boldsymbol{\lambda}_k,\boldsymbol{\tau}}$ for k such that $i \in \mathcal{A}_{\boldsymbol{\lambda}_k}$.

Theorem 5 (Errors in Correlation Ratios)**.** *Under Assumption 5 we find*

$$|\xi_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}} - \xi_{i,\boldsymbol{\lambda}_k,\boldsymbol{\tau}}| \leq \sqrt{\frac{1}{r_{i,\boldsymbol{\tau}}^2} - 1} \sqrt{\sum_{l=2}^{L} \rho_{\boldsymbol{\lambda}_k,l}^2} + \lambda_{i,k} \frac{\sigma^2(\epsilon_i)}{\sigma_i \, \sigma_{\boldsymbol{\lambda}_k}}. \tag{19}$$

where

$$\xi_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}} = \frac{1}{\rho_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}}} - \sum_{l=2}^{L} \left(\frac{\rho_{\boldsymbol{\lambda}_k,l}}{\rho_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}}} \right) \rho_{\boldsymbol{\lambda}_k,l} - \frac{1}{\rho_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}}} \sum_{i=1}^{I} \lambda_{i,k}^2 \frac{\sigma^2(\epsilon_i)}{\sigma_{\boldsymbol{\lambda}_k}^2}.$$

Clearly, these errors go to zero if the non-market factor loadings go to zero and the portfolio is well-diversified. For example, assume that equilibrium is described by two optimal portfolios equal to the above-mentioned low-beta portfolio and high-beta portfolio and assume further that the Fama-French three-factor model captures all relevant risk factors. For the low-beta portfolio, the market model yields a R-squared of 92.2 percent and the three-factor model explains 94.0 percent. This implies that the first right-hand-side term of (19) equals $\sqrt{(0.940/0.922)-1}\sqrt{0.940-0.922} = 0.019$. The portfolio is well-diversified and we can ignore the second right-hand side term. Similarly, the high-beta portfolio involves R-squared values of 94.5 and 95.4 percent and the absolute error in (19) is bounded by $\sqrt{(0.954/0.945)-1}\sqrt{0.954-0.945} = 0.009$. The error bounds will be even smaller if the two optimal portfolios overlap. The numerical example below further illustrates the goodness of the CSML approximation.

D Numerical example

We will now illustrate the CSML approximation with a numerical example that is constructed from historical U.S. stock market data. To allow for a compact presentation, we reduce the cross-section by constructing ten decile portfolios based on the past market-beta estimates of the individual stocks. Table I describes our procedures for data selection and portfolio formation and includes summary statistics.

We consider a simple case of our heterogeneous-investor model with one conservative investor ($\zeta_1 = 40$) and one adventurous investor ($\zeta_2 = 400$). Both investors are assumed to make combinations of the ten beta deciles and a riskless Treasury bill without riskless borrowing and short selling. Both investors are assumed to have the same wealth level ($w_1 = w_2 = 0.5$).

To obtain an ex-ante return distribution that is consistent with our two-investor model, we make small adjustments to the sample statistics of the historical data set. The standard deviations and market betas appear relatively stable in subsamples and, in addition, show a monotone increasing cross-sectional pattern, consistent with the past market betas used to create the deciles. Apparently, the portfolio formation procedure effectively creates stable risk profiles, and we therefore use the sample covariance matrix as the population covariance matrix. By contrast, the average returns appear relatively unstable in subsamples and do not show an increasing cross-sectional pattern. We therefore set our population expected returns by computing the smallest possible perturbation to the historical average returns (measured by the average absolute deviation) for which we can find two feasible portfolios that satisfy the equilibrium conditions: (i) the two portfolios are optimal and thus satisfy the KKT conditions (3) to (6), and (ii) the two portfolios aggregate to the market portfolio, or the value-weighted average of the ten deciles (Assumption 4).

Table I includes the optimal solution. Surprisingly small data perturbations are needed to obey the equilibrium conditions for this simple two-investor model: the average absolute error is less than three basis points per month. The conservative investor holds a combination of low-beta stocks (P_1) and the adventurous investor combines high-beta stocks (P_2). The medium-beta stocks in the fourth beta decile appear in both portfolios.

[Table I about here.]

Panel A of Figure 1 shows a mean-variance diagram with the expected returns and standard deviations of the ten beta deciles and the Treasury bill, together with the efficient frontier for the case without riskless borrowing and short sales. For the sake of comparison, we also show the unadjusted historical average returns and the associated efficient frontier. The latter case is not consistent with equilibrium because all efficient portfolios are based on the third, fifth and tenth decile and exclude the other deciles, preventing the market to clear. Our adjustments represent the smallest data perturbations consistent with our two-investor model. A representative-investor model would require much larger adjustments, because all deciles would then have to enter (with market weights) in a single efficient portfolio.

Panel B shows the full portfolio possibilities set (or all convex combinations of the base assets) and adds the optimal indifference curves and portfolios of the conservative and adventurous investors, together with the market portfolio.

The correlation between all possible stock portfolios is very high. For example, the correlation between the low-beta portfolio P_1 and the high-beta portfolio P_2 is higher than 90%, despite the minimal overlap between P_1 and P_2. Therefore, diversification between the deciles does not reduce variance in a material way, and, absent riskless borrowing and short selling, the investor adjusts the risk level of her portfolio by either combining low-beta stocks and Treasury bills and excluding high-beta stocks (in the case of low risk tolerance) or combining high-beta stocks and excluding low-beta stocks and Treasury bills (high risk tolerance). Paradoxically, diversifying across the optimal portfolios P_1 and P_2 produces inefficient portfolios. Most notably, the market portfolio, a weighted average of the two optimal portfolios, is dominated by medium-beta stocks. Clearly, the efficient set is not convex in this example.

Panel C and D illustrate the portfolio optimality conditions (3) to (6) for the two investors. For optimal portfolios, the included stocks exhibit a linear relation between their expected return and their beta relative to the optimal portfolio; the excluded stocks will lie below the line. For example, the conservative investor's portfolio (P_1) is characterized by a linear relation for the low-beta stocks, the steep slope reflecting the investor's low risk tolerance. The expected return of the high-beta stocks, which are excluded from P_1, is lower than predicted by this relation. This deviation does not reflect that P_1 is inefficient, but rather the positive shadow price of the restriction on short-selling high-beta stocks. Panel E shows that the betas relative to P_2 are nearly proportional to the betas relative to P_1, reflecting the very high correlation between the two portfolios. This near-proportionality explains why Panel D shows a very similar, kinked shape as Panel C; the two panels are nearly identical, apart from a scalar multiplication of the betas (and a different active set).

Panel F shows the expected returns and market betas for the ten deciles. By construction, these values obey the GSML relation (see Theorem 1), because the individual portfolios obey the optimality conditions and the market clears in this example. Since portfolios P_1 and P_2 are highly correlated with the market portfolio, the pattern is similar to that in Panel C and D. The dashed, straight line represents Black's (1972) general linear SML, or, in this case, the average of the two straight lines in Panel C and D. Clearly, the market portfolio does not obey the optimality conditions. It includes all stocks and therefore efficiency requires a linear relation. However, the expected return to medium-beta stocks is substantially higher than what is obtained through linear interpolation of the expected returns of low-beta stocks and high-beta stocks. The solid, kinked line represents our CSML approximation (see Theorem 3), which combines the two sets of optimality conditions under the assumption that P_1 and P_2 have zero non-market factor loadings (see Theorem 2).

Given that the optimal portfolios P_1 and P_2 are highly diversified and are highly correlated with the market portfolio, it is not surprising that this model gives a very good approximation (as Theorem 4 and 5 suggest); the average absolute error is smaller than 6.4 basis points per annum. The small errors primarily reflect small differences in the non-market factor loadings between the two optimal portfolios; the low-beta portfolio P_1 has a slightly lower SMB and HML loading than the high-beta portfolio P_2. A perfect fit would be achieved if we estimated the covariance matrix using a single-factor model rather than using the sample covariance matrix.

[Figure 1 about here.]

The market portfolio seems not far away from the efficient frontier in economic terms. Increasing the expected return of the average portfolio by about 40 basis points per annum would make the average portfolio efficient. In this example, the small distance from the frontier in our case primarily reflects the use of a compact portfolio possibilities set of convex combinations of ten highly diversified and correlated base assets (the decile portfolios) with an average return spread of only 479 basis points per annum. The short distance to the frontier serves as a reminder that relatively large deviations from the linear SML can be accompanied by relatively small deviations from the frontier. Despite the short distance, relatively large data perturbations are required to make the market portfolio efficient and create a linear mean-beta relationship in this example. To achieve efficiency and linearity, we cannot simply increase the expected market return by 40 basis points per annum, but must lower the expected return of the low-beta stocks and increase the expected return of the high-beta stocks by much larger amounts.

II Empirical Analysis

Empirical tests of the CAPM broadly fall into two categories: (i) structural tests of mean-variance efficiency of the market portfolio and (ii) regression analysis of the linear cross-sectional relationship between expected return and market beta (SML). Most formal empirical tests in the first category focus on the intercepts in time-series regressions of asset returns on the market index in the spirit of Black, Jensen, and Scholes (1972). Representative methods include the multivariate tests developed by Gibbons, Ross, and Shanken (1989), the GMM test conducted by MacKinlay and Richardson (1991), and the Bayesian inference pursued by Kandel, McCulloch, and Stambaugh (1995) and Wang (1998). Tests in the second category build on the classic two-pass regression procedure of Fama and MacBeth (1973), which is used nowadays not only in asset pricing but also in many other areas of finance, accounting and economics. Shanken (1992), Jagannathan and Wang (1998) and Shanken and Zhou (2007) provide important discussions of the statistical properties and performance of this approach.

For several reasons, this study emphasizes the two-pass regression method. This approach has more flexibility to analyze and test the functional form of the SML. The efficiency tests use the null hypothesis that the market portfolio is efficient, or, equivalently, the SML is linear. By contrast, our null hypothesis is that different individual investors hold different efficient portfolios that are dominated by market risk and that combine to yield a concave SML. Portfolio efficiency tests can test this null only if we specify the relevant set of portfolio restrictions, the composition of the individual portfolios and the distribution of wealth. In addition, the two-pass regression method has the flexibility to control for the effect of empirically relevant co-variates (that do not enter in structural models), such as a stock's non-systematic risk and market capitalization.

This section applies the regression method to a set of 100 stock portfolios that are based on the market capitalization and market beta of individual stocks, as well as to the entire cross-section of individual stocks (Section IIID).

A Data

We use monthly total stock returns from the Center for Research in Security Prices (CRSP) and the one-month US Treasury bill rate from Ibbotson and Associates. Following the convention, our analysis focuses on ordinary common U.S. stocks listed on the New York Stock Exchange (NYSE), American Stock Exchange (AMEX) and Nasdaq markets, excluding ADRs, REITs, closed-end-funds, units of beneficial interest, and foreign stocks. We require stocks to have at least 24 observations available in the past 60 months for estimating the market betas. A stock is excluded from the analysis if price information is no longer available. In that case, the delisting return or partial monthly return provided by CRSP is used as the last return observation. Our analysis is based on the 2010 edition of CRSP data set and we employ the full time-series from January 1926 to December 2010.

The market betas of individual stocks can be difficult to estimate with high accuracy, due to relatively high stock-specific risk, and potential structural and cyclical changes over time. It is well known that these problems can be mitigated by forming portfolios of stocks and periodically re-balancing these portfolios, a tradition that goes back to Blume (1970). It is particularly useful to form "beta portfolios" based on the past market beta estimates of individual stocks, as in, for example, Fama and MacBeth (1973). This approach yields relatively accurate and stable market beta estimates and a wide spread in the market betas, increasing the statistical power of the analysis. Furthermore, a stable market beta avoids the possible bias that arises if market beta is correlated over time with the market-risk premium, and the associated need to use a conditional model specification; see, for example, Jagannathan and Wang (1996) for a discussion of this problem.

In practice, market beta is strongly negatively correlated with market capitalization of equity ("size"), and, in addition, small-cap stocks appear to carry a return premium that is uncorrelated with their market betas. To disentangle these two effects, we apply a double-sorting routine based on market capitalization and market beta, following Fama and French (1992).

At the end of each month, all stocks that fulfill our data requirements are sorted based on their market capitalization and divided into ten segments using NYSE size-decile breakpoints. Next, each size segment is further divided into ten beta-deciles based on the past 60-month beta estimates of the individual stocks. The result is 100 test portfolios with independent variation in market capitalization and market beta. The portfolio formation starts in December 1927, 24 months after the beginning of the CRSP files, because at least 24 months of prior data are needed for estimating the betas of the individual stocks.

In order to avoid the well-known phenomenon of clustering negative (positive) sampling errors in beta estimates for low-beta (high-beta) portfolios, we re-estimate individual stock betas over the following 60-month period. Following Fama and MacBeth (1973), we then aggregate individual stock betas to form portfolio betas and use these post-formation portfolio betas to predict out-of-sample portfolio returns. A case in point is the prediction of portfolio returns for January 2010. The relevant portfolios are formed on December 31, 2004, using the stocks' market caps and past 60-month market betas based on stock returns from January 2000 to December 2004. The betas of the individual stocks in the subsequent 60-month period from January 2005 to December 2009 are then aggregated to form portfolio betas. These portfolio beta estimates are used to predict the portfolios' returns in January 2010. For each test portfolio, portfolio betas and returns are computed as value-weighted averages of individual stock betas and returns.

A possible drawback of the double-sorting routine is that the 10 size segments differ substantially with respect to their economic and statistical significance. Most notably, the first two size segments of micro-cap stocks represent about 1.2 percent of the total market capitalization in the average month in our sample. Relatively high transaction costs and low liquidity further reduce the economic relevance of this market segment. In addition, data for this market segment are known to be relatively noisy and contain anomalous patterns that seem to defy rational explanation. For these reasons, micro-cap stocks are often excluded from the analysis or analyzed separately; see, for example, Fama and French (2008). Following this tradition, we will analyze the robustness of our results and conclusions for excluding the first two size segments, leaving 80 test portfolios (8x10). In addition, we repeat all of our empirical analysis using individual stocks, both including and excluding the micro capitalization segment.

Our main analysis is based on the period from January 1933 to December 2010, drawing also on the period from January 1926 to December 1932 for portfolio formation and beta estimation at the start of the sample period. This long sample period seems particularly useful, because it includes the bear market of the 1930s in addition to the bear markets of the 1970s and 2000s. The data prior to July 1963 is sometimes excluded from the analysis to avoid known biases associated with the Compustat database, which is the usual source of data about the book value of equity. We complement the pre-1963 Compustat book equity data with hand-collected book equity values from the Moody's Industrial, Public Utility, Transportation, and Bank and Finance Manuals - made publicly available on Ken French's web-page. In addition, we analyze the robustness of our results in the later period from July 1963 to December 2010, to avoid possible concerns about the quality of the data and to facilitate a comparison with many other studies that focus on the post-1963 sample.

B Methodology

Our analysis employs the two-pass regression methodology of Fama and MacBeth (1973). The analysis is build around their original (second-pass) cross-sectional regression equation:

$$x_{p,t} - r_{F,t} = \Gamma_{0t} + \Gamma_{1t}\,\hat{\beta}_{p,t-1} + \Gamma_{2t}\,(\hat{\beta}_{p,t-1})^2 + \Gamma_{3t}\,\hat{\sigma}_{p,t-1}(\epsilon_p) + \kappa_{p,t}. \tag{20}$$

In this equation, $\hat{\beta}_{p,t-1}$ is the value-weighted average of the estimated beta ($\hat{\beta}_{i,t-1}$) of stock i in portfolio p and $\hat{\sigma}_{p,t-1}(\epsilon_p)$ is the value-weighted average of the residual standard deviation, $\hat{\sigma}_i(\epsilon_i)$. Our results are robust to using equal-weighted averages and to using the squared average beta rather than the average squared beta. The chosen specification however better aligns with our theoretical model. The regressors are lagged by one month relative to the test month in which the portfolio excess return is measured.

The regression model uses beta-squared to measure non-linearity with a single parameter. Our theoretical model predicts a general concave, piece-wise linear relation with a different linear line segment for every investor. It is difficult to implement this functional form in practice, because it requires the specification of the relevant number of linear pieces and the associated return intervals. One possible specification is a four-piece linear function with kinks at the first, second and third quartile of the cross-sectional beta distribution. Interestingly, this model yields results that are very similar to those based on beta-squared.

Beta and beta-squared are strongly correlated and the question arises whether structural multi-collinearity affects the estimates for the individual regression coefficients and their standard errors. Although isn't immediately clear that multi-collinearity will have strong effects on a two-pass regression analysis, we use the following centered regression model to mitigate any possible effects:

$$x_{p,t} - r_{F,t} = \gamma_{0t} + \gamma_{1t}\,\hat{\beta}_{p,t-1} + \gamma_{2t}\,(\hat{\beta}_{p,t-1} - \bar{\beta}_{t-1})^2 + \gamma_{3t}\,\hat{\sigma}_{p,t-1}(\epsilon_p) + \eta_{p,t}. \tag{21}$$

where $\bar{\beta}_{t-1}$ is the sample average beta. Whereas beta has a correlation with beta-squared of about 97%, the correlation with the centered-beta-squared is only about 16% for the full-sample of 100 size-beta portfolios. For the economic interpretation, the original, non-centered parameters can be recovered from the centered parameters in the following manner:

$$\begin{aligned}
\Gamma_0 &= \gamma_0 + \gamma_2\,\bar{\beta}_{t-1}^2, \\
\Gamma_1 &= \gamma_1 - 2\,\gamma_2\,\bar{\beta}_{t-1}, \\
\Gamma_2 &= \gamma_2, \\
\Gamma_3 &= \gamma_3.
\end{aligned}$$

Centering does not affect the joint significance of beta and beta-squared, the regression results for residual risk (and possible other regressors) or the overall goodness of fit. The purpose of centering in our analysis is to avoid the possibility of inflating the individual Fama-MacBeth t-statistics of beta and beta-squared. As discussed below, we will also explicitly measure the joint significance of beta and beta-squared.

The cross-sectional regression is estimated every month using Ordinary Least Squares regression analysis to generate a time-series of monthly coefficient estimates. The time-series averages and standard deviations of the coefficient estimates are used for testing hypotheses regarding the unknown parameters γ_{0t}, γ_{1t}, γ_{2t} and γ_{3t}. The monthly coefficient estimates may show patterns of positive or negative autocorrelation that can cause the original Fama-McBeth standard errors to be biased. This study corrects for autocorrelation by using a Newey and West (1987) correction with 12 monthly lags. In our study, the adjustment has a negligible effect on the standard errors for all regressors. Our results are also robust to explicit corrections for cross-serial correlation and heteroskedasticity. For example, we find similar results for beta and beta-squared with the cluster-robust method that is analyzed by Petersen (2009) or a full-fledged mixed model of fixed-random effects panel method. We therefore, report only the standard Fama-McBeth results (with a Newey-West correction) in this study. The methodological robustness of the results is perhaps unsurprising given that our data set fulfills the conditions that Petersen (2009) identified as favorable for the Fama-MacBeth method, namely a sample in which the autoregressive pattern dies off fast and the time series is long.

The hypothesis of Fama and MacBeth (1973) is that there exists a strictly positive and linear cross-sectional relation between expected return on the one hand and market beta on the other hand ($\gamma_{1t} > 0$ and $\gamma_{2t} = 0$). Our alternative hypothesis is that there exists a concave relation between these variables; the coefficient for market beta is strictly positive and the coefficient of beta-squared is strictly negative ($\gamma_{1t} > 0$ and $\gamma_{2t} < 0$). Since we allow riskless lending but exclude riskless borrowing, the intercept is predicted to be non-negative ($\gamma_{0t} \geq 0$). However, the inclusion of other stock characteristics as additional regressors can cause negative values for the intercept (if these regressors are not de-meaned). The coefficient of residual risk is predicted to be zero ($\gamma_{3t} = 0$). Nevertheless, a non-zero coefficient value may arise if residual risk operates as a proxy for market beta (given that market beta is estimated with error), or, alternatively, as a proxy for other, omitted stock characteristics (see below). For instance, Ang *et al.* (2006) report a strong negative relation between average stock returns and residual risk, a result that is challenged Bali and Cakici (2008).

Apart from the individual regression coefficients, we also analyze the "average beta premium" (ABP), or the predicted beta-risk premium for a stock or stock portfolio with a market beta equal to one: $ABP_t = \Gamma_{1t} + \Gamma_{2t} = \gamma_1 + \gamma_2\,(1 - 2\,\bar{\beta})$. Given that the market portfolio has a market beta of one, the value of the ABP is expected to be close to the historical excess return to the market portfolio. ABP is also relevant to address possible concerns about multi-collinearity (that may remain after centering); is a single parameter that captures the joint effect of beta and beta-squared.

Empirical research on stock returns has identified several other relevant stock characteristics in addition to market beta and residual risk; see, for example, Basu (1977, 1983), Banz (1981), De Bondt and Thaler (1985), Jegadeesh (1990), Jegadeesh and Titman (1993), Acharya and Pedersen (2005). Our analysis will include popular corrections for market capitalization of equity (ME), book-to-market equity ratio (BtM), short-term reversal (R_1), momentum (R_{12-2}) and long-term reversal (R_{60-13}) and illiquidity ratio ($Illiq$). It seems essential to include market capitalization as a regressor, because market beta and residual risk are strongly negatively correlated with market capitalization; the average small-cap stock has a relatively high market beta and residual risk. By contrast, the other return variables show a weaker correlation with market beta and residual risk, and most of this correlation stems from a joint correlation with market capitalization. We do not question the explanatory power of these characteristics for average returns; we merely claim that they do not materially affect the explanatory power of beta and beta-squared in our samples of size-beta portfolios after controlling for market capitalization. Similarly, the inclusion of beta-squared is unlikely to affect the explanatory power of these characteristics.

For a given test portfolio, ME represents the logarithm of the total market capitalization of firms in billions of dollars prior to the current month, and BtM is the logarithm of the median book to market value of equity. The book value of equity is defined as in Fama and French (1996). $Illiq$ is the logarithm of the prior 12-month moving average of median monthly Amihud's (2002) illiquidity ratio, where dollar trading volume is defined in billions of dollars. Value weighted averages of past stock returns over the first, second through 12th, and 13th through 60th months prior to the current month are aggregated to form portfolio past returns variables. Following Brennan, Chordia, and Subrahmanyam (1998), R_1, R_{12-2} and R_{60-13} are equal to the logarithms of the cumulative past gross returns, respectively.

A logarithmic transformation is applied for several regressors in order to allow for a diminishing effect and/or to mitigate the effect of outliers. These considerations are particularly relevant when analyzing individual stocks (rather than stock portfolios), given the relatively wide sample range and high noise level in this case. A logarithmic transformation for market beta is however unusual, as the standard theory explicitly predicts a linear SML, and, in addition, the sample range of beta is relatively narrow. Although our theory predicts a concave SML, we do not consider a logarithmic specification suitable for our analysis. First, for testing hypotheses, it is desirable to use a specification that includes the linear SML as a special case. Second, the logarithmic specification allows for the risk premium to decrease with the beta, but it also assumes that the effect is strongest for low-beta stocks, while the data suggest otherwise. Indeed, if log beta is used instead of beta in our regressions, the empirical fit improves, but log-beta-squared remains significant.

For the sake of brevity, we do not report estimation results for the squared terms of regressors other than market beta. Neither our theoretical analysis nor the empirical literature assigns a role to, for instance, squared residual risk or squared market capitalization; in fact, these regressors are deliberately defined as concave transformations of an underlying variable (the logarithm of market capitalization and the square root of residual variance, for instance). Indeed, none of the additional squared terms has a significant effect or changes the results for beta-squared in a material way in our analysis.

C Main Results

Table II summarizes our results for the entire cross-section of 100 size-beta portfolios and the full time-series of monthly portfolio returns from January 1933 to December 2010. The top panel shows regressions that exclude beta-squared and residual risk. The univariate, beta-only regression yields a positive beta risk premium of 28.6 basis points per month, or 3.43 percent per annum. The Fama-MacBeth t-statistic of 1.49 means that the coefficient is (at best) marginally significant. By contrast, the average Adjusted R-squared of about 13 percent, suggesting that market beta does plays a significant role in the individual months. The difference between the two statistical goodness measures partly reflects time-variation in the coefficient value: the coefficient takes substantially different, but significant, values in different months. In addition, the R-squared is inflated by the cross-sectional correlation and heteroskedasticity of the SML error terms, whereas the Fama-MacBeth t-statistic avoides the use of cross-sectional standard errors. Including market capitalization as a second regressor further reduces the coefficient estimate and the t-statistic of market beta and yields a significantly negative coefficient for Size, confirming the results of Banz (1981), Fama and French (1992), and many others. The beta risk premium estimate further decreases somewhat after including BtM or $Illiq$ and it increases somewhat after including the past return variables R_1, R_{12-2} and R_{60-13}, but it remains economically and statistically insignificant in all cases. Consistent with what is documented elsewhere in the empirical literature, we find significantly negative premiums for ME, R_1 and R_{60-13}, and significantly positive premiums for BtM and R_{12-2}. The effect of $Illiq$ is confounded with the effect of ME in our analysis, but it has a significantly positive premium if we remove ME as a regressor.

The conclusions change considerably if we add beta-squared to the regression. The center panel reveals a significant and concave relationship between return and beta. Since we use the square of the centered beta, $(\hat{\beta}_{t-1} - \bar{\beta}_{t-1})^2$, to avoid multi-collinearity problems, the linear beta coefficient resembles the beta premium of the beta-only model and is not informative here. The "average beta premium" (ABP) is estimated to be 0.949, or 11.39 percent per annum, more than three times the value in the beta-only regression and highly significant (t-statistic: 3.54). Consistent with our hypothesis, the coefficient estimate for beta-squared is significantly negative (t-statistic: -3.63), and the risk premium increases at a diminishing rate. Whereas including market capitalization weakens the linear return-risk relation (see the top panel), it has no material effect on the concave relation and the estimated ABP. Adding the other stock characteristics also does not materially change the results and conclusions about beta and beta squared. For example, the last row of the center panel shows a statistically significant ABP of 0.607, or about seven percent per annum. Interestingly, the average beta premium ranges from about seven percent to 11 percent per annum in all model specifications, enveloping the historical equity premium of about 8.5 percent in this period.

The bottom panel adds residual risk as an additional regressor. In the first regression, the coefficient of residual risk is estimated to be 18.9 basis points per month, or 2.74 percent per annum. Residual risk appears to have significant explanatory power, but it does not materially affect the results and conclusions about beta and beta-squared; the return-beta relation remains significantly concave and the estimated ABP even increases slightly. In addition, the inclusion of market capitalization drives out the role of residual risk. These results are supportive of our model, which builds on systematic risk and does not assign a role to residual risk. The weak role of residual risk may reflect that we analyze size-beta portfolios rather than individual stocks, and that the estimated betas of these portfolios are more accurate and stable than the estimated betas of individual stocks. These results confirm the conclusion of Bali and Cakici (2008) that there exists no robust, significant relation between average stock returns and residual risk.

Interestingly, the estimated ABP ranges from about seven percent to 11 percent per annum in all model specifications, enveloping the historical equity premium of about 8.5 percent in this period.

[Table II about here.]

We have thus far analyzed the full time-series of returns from January 1933 to December 2010 and the full cross-section of 100 size-beta portfolios. The question arises whether our results are robust to the sample period and cross-section under consideration. Table III shows that our results and conclusions are robust to excluding the early period from January 1933 to June 1963 and/or micro-cap stocks from the sample. The most notable effect is a modest reduction in the estimated ABP after the exclusion of micro-caps in the full sample period. The return-risk relationship however remains significantly concave and the estimated ABP remains firmly above seven percent per annum in every subsample and every model specification. By contrast, the role of ME and BtM is reduced to marginal significance after the exclusion of the early sample period and micro-cap stocks.

[Table III about here.]

D Stock-Level Results

As discussed above, there exist several compelling arguments for analyzing size-beta portfolios rather than individual stocks. Nevertheless, a possible concern is that the sorting of stocks and the formation of portfolios affects the statistical size and power of the analysis. For example, sorting stocks based on ME and beta can produce artificially high sample variation for the portfolio values of those characteristics and artificially low sample variation for other characteristics. To address this concern, this section analyzes the cross-section of individual stocks rather than portfolios.

Table IV summarizes our results for the entire cross-section of individual stocks and the full time-series of monthly excess returns from January 1933 to December 2010. The results are remarkably similar to those in Table II for size-beta portfolios. Again, a significant, robust and concave return-beta relationship appears. The estimated ABP is economically and statistically significant, although at a lower level compared to the portfolio analysis, varying from around five percent to above six percent per annum. A notable distinction is the higher residual risk premium. In our individual stock sample, the correlation between residual risk and the standard error of beta estimate, a measure of accuracy, is well above 90 percent. It therefore seems plausible that the residual risk in our specifications picks up the higher noise for the individual stock beta estimates. As seen elsewhere in the empirical asset pricing literature, we find significantly negative premiums for ME, R_1 and R_{60-13}, and significantly positive premiums for BtM and R_{12-2}. Perhaps not surprisingly, the short-term reversal effect (R_1) is much stronger for individual stocks than for portfolios.

[Table IV about here.]

Table V shows that our stock-level results and conclusions are also robust to excluding the early period from January 1933 to June 1963 and/or micro-cap stocks from the analysis. The return-risk relationship remains significantly concave and the average risk premium hovers around five to six percent per annum in every subsample and every model specification.

[Table V about here.]

III Conclusions

Binding investment restrictions generally distort the classical linear relation between expected return and market beta (SML) in a systematic way. In this case, the expected return to a given risky security will reflect the relative risk tolerance of investors who include the security in their optimal portfolios and the covariance of the security with those portfolios. To the extend that the optimal portfolios are correlated and share a common exposure to market risk, expected return will tend to be a concave function of the traditional market beta (rather than the traditional linear function). Indeed, a high positive correlation and substantial joint exposure to market risk is typical for the portfolios of many mutual funds and institutional investors, even if their portfolio composition shows large differences. The concave relation arises in the framework of Sharpe (1991) with mean-variance preferences and without short-selling and borrowing, but also for more general preferences and restrictions (see the discussion in Section IA).

Sharpe (1991) concludes that advances in financial technology and knowledge relax investment constraints and will move markets closer the assumptions of the CAPM. However, closeness to the assumptions is not sufficient for closeness to the equilibrium conditions. Restrictions tend to make the individual portfolios more similar and relaxations introduce more possibilities for differences between individual portfolios. In addition, relaxations generally will not affect all investors in the same way. Many institutional investors and mutual funds face legal or contractual constraints on borrowing and short selling that cannot be relaxed by means of clever financial engineering. In such situations, financial innovation may in effect increase differences between investors' active sets. For these reasons, it is not obvious that financial innovation moves the market closer to a linear SML.

The pattern of expected returns ultimately seems an empirical issue. Our empirical analysis reveals a significant and robust, concave relation between average return and estimated market beta for stocks, consistent with our hypothesis. The inclusion of beta-squared in the regression yields a beta-risk premium of at least five to six percent for the average stock (with a market beta of one), substantially higher than conventional estimates and statistically highly significant. In addition, beta-squared has a significantly negative coefficient, implying that the risk premium increases at a diminishing rate. Encouragingly, the role of concavity is robust to the inclusion of other stock characteristics and the selection of the cross-section and sample period. Concavity appears not only in the analysis of individual stocks but also for aggregated size-beta portfolios, which reduces concerns about estimation error and time-variation of stock-level betas. Similar results are found when using alternative functional forms and using a variety of regression methods.

These empirical findings confirm our theoretical analysis and contrast with the empirical results of Fama and MacBeth (1973), who conclude that beta-squared plays no significant role and that the SML appears linear. A closer look at the original results of Fama and MacBeth (1973, p. 623, Table 3D) reveals that the coefficient of beta-squared is actually significantly negative in the only sub-period that shows a significant role for market beta (1946-1955). However, beta-squared is not significant in their full sample period, 1935-1968, which is comparable to our early sub-period, 1933-1963. A further analysis reveals that the different results and conclusions can be explained by differences in the portfolio construction procedure. The original study uses twenty beta-portfolios, without controlling for market capitalization, making it difficult to disentangle the competing effects of market beta and market capitalization. Most notably, (single-sorted) high-beta portfolios tend to include a disproportional number of micro-cap and small-cap stocks, and these portfolios benefit from the return premium that these stocks earn (independent of their risk levels), obscuring the underlying mean-beta relationship. Analyzing size-beta portfolios or individual stocks introduces more independent variation in market beta and market capitalization and allows for disentangling the two competing effects.

A Proofs

A Proof of Theorem 1

Multiplying the optimality condition (3) for investor k by ζ_k implies

$$\zeta_k\, \mu_i = \mathrm{Cov}\left(x_i, \mathbf{x}^T \boldsymbol{\lambda}_k\right) + \zeta_k\, \theta_{\boldsymbol{\lambda}_k} - \zeta_k\, \alpha_{i,\boldsymbol{\lambda}_k} \tag{22}$$

for all $i \in \mathcal{I}$ and all $k \in \mathcal{K}$. Aggregating (22) across investors $k \in \mathcal{K}$ using wealth shares $w_k/(1+\lambda_{N+1,k})$ invested in the risk securities yields

$$\sum_{k\in\mathcal{K}} (w_k/(1+\lambda_{N+1,k}))\, \zeta_k\, \mu_i = \sum_{k\in\mathcal{K}} (w_k/(1+\lambda_{N+1,k})) \left(\mathrm{Cov}\left(x_i, \mathbf{x}^T \boldsymbol{\lambda}_k\right) + \zeta_k\, \theta_{\boldsymbol{\lambda}_k} - \zeta_k\, \alpha_{i,\boldsymbol{\lambda}_k}\right)$$

which is equivalent to

$$\bar{\zeta}\, \mu_i = \mathrm{Cov}\left(x_i, \mathbf{x}^T \boldsymbol{\tau}\right) + \bar{\zeta}\, \bar{\theta} - \bar{\zeta}\, \bar{\alpha}_i. \tag{23}$$

Replacing $\mathrm{Cov}\left(x_i, \mathbf{x}^T \boldsymbol{\tau}\right)$ with $\beta_{i,\boldsymbol{\tau}}\, \sigma^2_{\boldsymbol{\tau}}$ and dividing by $\overline{\zeta}$ implies (11). □

B Proof of Theorem 2

Under Assumption 5, we find:

$$\xi_{i,\boldsymbol{\lambda}_k,\boldsymbol{\tau}} = \rho_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}} + \sum_{l=2}^{L} \left(\frac{\rho_{i,l}}{\rho_{i,\boldsymbol{\tau}}}\right) \rho_{\boldsymbol{\lambda}_k,l} + \lambda_{i,k}\, \frac{\sigma^2(\epsilon_i)}{\sigma_i\, \sigma_{\boldsymbol{\lambda}_k}}. \tag{24}$$

Using

$$\frac{1}{\rho_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}}} = \rho_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}} + \frac{1-\rho^2_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}}}{\rho_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}}}.$$

and

$$1 = \rho_{\boldsymbol{\lambda}_k,\boldsymbol{\lambda}_k} = \rho^2_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}} + \sum_{l=2}^{L} \rho^2_{\boldsymbol{\lambda}_k,l} + \sum_{i=1}^{I} \lambda^2_{i,k}\, \frac{\sigma^2(\epsilon_i)}{\sigma^2_{\boldsymbol{\lambda}_k}}$$

we obtain

$$\frac{1}{\rho_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}}} = \rho_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}} + \sum_{l=2}^{L} \left(\frac{\rho_{\boldsymbol{\lambda}_k,l}}{\rho_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}}}\right) \rho_{\boldsymbol{\lambda}_k,l} + \frac{1}{\rho_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}}} \sum_{i=1}^{I} \lambda^2_{i,k}\, \frac{\sigma^2(\epsilon_i)}{\sigma^2_{\boldsymbol{\lambda}_k}}.$$

It follows that

$$\rho_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}} = \frac{1}{\rho_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}}} - \sum_{l=2}^{L} \left(\frac{\rho_{\boldsymbol{\lambda}_k,l}}{\rho_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}}}\right) \rho_{\boldsymbol{\lambda}_k,l} - \frac{1}{\rho_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}}} \sum_{i=1}^{I} \lambda_{i,k}^2 \frac{\sigma^2(\epsilon_i)}{\sigma^2_{\boldsymbol{\lambda}_k}}.$$

Placing this equation in (24) gives

$$\xi_{i,\boldsymbol{\lambda}_k,\boldsymbol{\tau}} = \xi_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}} + \sum_{l=2}^{L} \left(\frac{\rho_{i,l}}{\rho_{i,\boldsymbol{\tau}}}\right) \rho_{\boldsymbol{\lambda}_k,l} + \lambda_{i,k} \frac{\sigma^2(\epsilon_i)}{\sigma_i \, \sigma_{\boldsymbol{\lambda}_k}} \tag{25}$$

where

$$\xi_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}} = \frac{1}{\rho_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}}} - \sum_{l=2}^{L} \left(\frac{\rho_{\boldsymbol{\lambda}_k,l}}{\rho_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}}}\right) \rho_{\boldsymbol{\lambda}_k,l} - \frac{1}{\rho_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}}} \sum_{i=1}^{I} \lambda_{i,k}^2 \frac{\sigma^2(\epsilon_i)}{\sigma^2_{\boldsymbol{\lambda}_k}}.$$

Finally, using $\rho_{\boldsymbol{\lambda}_k,l} = \mathbf{b}_l^T \boldsymbol{\lambda}_k \, \sigma_{f_l}/\sigma_{\boldsymbol{\lambda}_k} = 0$ for all $l = 2, \ldots, L$ and $\lambda_{i,k}\, \sigma^2(\epsilon_i) \to 0$ for all $i \in \mathcal{A}_{\boldsymbol{\lambda}_k}$ and $\lambda_{i,k} = 0$ for all $i \notin \mathcal{A}_{\boldsymbol{\lambda}_k}$, Equation (25) implies

$$\xi_{i,\boldsymbol{\lambda}_k,\boldsymbol{\tau}} = \xi_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}}$$

for all $i = 1, \ldots, N$ and

$$\xi_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}} = \frac{1}{\rho_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}}}.$$

C Proof of Theorem 3

The optimality conditions (3)-(6) imply the following set of inequalities and equalities

$$\begin{cases} \mu_i \leq \left(\frac{1}{\zeta_k}\, \sigma^2_{\boldsymbol{\lambda}_k}\right) \beta_{i,\boldsymbol{\lambda}_k} + \theta_{\boldsymbol{\lambda}_k} & \forall i \in \mathcal{I} \text{ and } k \in \mathcal{K} \text{ such that } i \notin \mathcal{A}_{\boldsymbol{\lambda}_k}, \\ \mu_i = \left(\frac{1}{\zeta_k}\, \sigma^2_{\boldsymbol{\lambda}_k}\right) \beta_{i,\boldsymbol{\lambda}_k} + \theta_{\boldsymbol{\lambda}_k} & \forall i \in \mathcal{I} \text{ and } k \in \mathcal{K} \text{ such that } i \in \mathcal{A}_{\boldsymbol{\lambda}_k}. \end{cases} \tag{26}$$

The portfolio beta

$$\beta_{i,\boldsymbol{\lambda}} = \frac{\mathrm{Cov}(x_i, x^T \boldsymbol{\lambda})}{\sigma^2_{\boldsymbol{\lambda}}} = \rho_{i,\boldsymbol{\lambda}} \frac{\sigma_i}{\sigma_{\boldsymbol{\lambda}}}$$

relative to risky portfolio $\boldsymbol{\lambda} \in \Lambda$ can be rewritten in terms of the market beta:

$$\beta_{i,\boldsymbol{\lambda}_k} = \frac{\sigma_{\boldsymbol{\tau}}}{\sigma_{\boldsymbol{\lambda}_k}} \frac{\rho_{i,\boldsymbol{\lambda}_k}}{\rho_{i,\boldsymbol{\tau}}} \beta_{i,\boldsymbol{\tau}} = \frac{\sigma_{\boldsymbol{\tau}}}{\sigma_{\boldsymbol{\lambda}_k}} \xi_{i,\boldsymbol{\lambda}_k,\boldsymbol{\tau}} \, \beta_{i,\boldsymbol{\tau}} \quad \forall i = 1, \ldots, N, \; k \in \mathcal{K}. \tag{27}$$

Under Assumption 6, $\xi_{i,\boldsymbol{\lambda}_k,\boldsymbol{\tau}} = \xi_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}}$ for all $i \in \mathcal{A}_{\boldsymbol{\lambda}_k}$ and $\xi_{i,\boldsymbol{\lambda}_k,\boldsymbol{\tau}} \leq \xi_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}}$ for all $i \notin \mathcal{A}_{\boldsymbol{\lambda}_k}$, and, therefore, (26) can be written as

$$\begin{cases} \mu_i \leq \left(\frac{1}{\zeta_k}\,\sigma_{\boldsymbol{\lambda}_k}\,\sigma_{\boldsymbol{\tau}}\,\xi_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}}\right)\,\beta_{i,\boldsymbol{\tau}} + \theta_{\boldsymbol{\lambda}_k} \quad \forall i = 1,\ldots,N \text{ and } k \in \mathcal{K} \text{ such that } i \notin \mathcal{A}_{\boldsymbol{\lambda}_k}, \\ \\ \mu_i = \left(\frac{1}{\zeta_k}\,\sigma_{\boldsymbol{\lambda}_k}\,\sigma_{\boldsymbol{\tau}}\,\xi_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}}\right)\,\beta_{i,\boldsymbol{\tau}} + \theta_{\boldsymbol{\lambda}_k} \quad \forall i = 1,\ldots,N \text{ and } k \in \mathcal{K} \text{ such that } i \in \mathcal{A}_{\boldsymbol{\lambda}_k}. \end{cases} \tag{28}$$

In equilibrium any risky security $i = 1,\ldots,N$ is included in at least some portfolio, that is, $i \in \mathcal{A}_{\boldsymbol{\lambda}_k}$, for some $k \in \mathcal{K}$. Therefore, aggregating (28) across investors $k \in \mathcal{K}$ yields

$$\mu_i = \min_{k \in \mathcal{K}} \left[\left(\frac{1}{\zeta_k}\,\sigma_{\boldsymbol{\lambda}_k}\,\sigma_{\boldsymbol{\tau}}\,\xi_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}}\right)\,\beta_{i,\boldsymbol{\tau}} + \theta_{\boldsymbol{\lambda}_k}\right].$$

The slope of the relation between μ_i and $\beta_{i,\boldsymbol{\tau}}$ is positive since $\xi_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}} \geq 0$. The minimum over increasing linear functions is always an increasing and concave function. □

D Proof of Theorem 4

The CSML (17) can be written as

$$\hat{\mu}_i = \min_{k\,:\,i \in \mathcal{A}_{\boldsymbol{\lambda}_k}} \left[\left(\frac{1}{\zeta_k}\,\sigma_{\boldsymbol{\lambda}_k}\,\sigma_{\boldsymbol{\tau}}\,\xi_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}}\right)\,\beta_{i,\boldsymbol{\tau}} + \theta_{\boldsymbol{\lambda}_k}\right].$$

Moreover, inspection of the proof of Theorem 3 shows that the GSML (11) is equivalent to

$$\mu_i = \min_{k\,:\,i \in \mathcal{A}_{\boldsymbol{\lambda}_k}} \left[\left(\frac{1}{\zeta_k}\,\sigma_{\boldsymbol{\lambda}_k}\,\sigma_{\boldsymbol{\tau}}\,\xi_{i,\boldsymbol{\lambda}_k,\boldsymbol{\tau}}\right)\,\beta_{i,\boldsymbol{\tau}} + \theta_{\boldsymbol{\lambda}_k}\right].$$

It follows that

$$\hat{\mu}_i - \mu_i = \min_{k\,:\,i \in \mathcal{A}_{\boldsymbol{\lambda}_k}} \left[\left(\frac{1}{\zeta_k}\,\sigma_{\boldsymbol{\lambda}_k}\,\sigma_{\boldsymbol{\tau}}\,\xi_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}}\right)\,\beta_{i,\boldsymbol{\tau}} + \theta_{\boldsymbol{\lambda}_k}\right] - \min_{k\,:\,i \in \mathcal{A}_{\boldsymbol{\lambda}_k}} \left[\left(\frac{1}{\zeta_k}\,\sigma_{\boldsymbol{\lambda}_k}\,\sigma_{\boldsymbol{\tau}}\,\xi_{i,\boldsymbol{\lambda}_k,\boldsymbol{\tau}}\right)\,\beta_{i,\boldsymbol{\tau}} + \theta_{\boldsymbol{\lambda}_k}\right]$$

and therefore

$$
\begin{aligned}
|\hat{\mu}_i - \mu_i| &\leq \max_{k\,:\,i\in\mathcal{A}_{\boldsymbol{\lambda}_k}} \left| \left(\frac{1}{\zeta_k}\, \sigma_{\boldsymbol{\lambda}_k}\, \sigma_{\boldsymbol{\tau}} \right) \, (\xi_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}} - \xi_{i,\boldsymbol{\lambda}_k,\boldsymbol{\tau}})\; \beta_{i,\boldsymbol{\tau}} \right| \\
&= \max_{k\,:\,i\in\mathcal{A}_{\boldsymbol{\lambda}_k}} \left| \left(\frac{\mu_{\boldsymbol{\lambda}_k} - \theta_{\boldsymbol{\lambda}_k}}{\sigma_{\boldsymbol{\lambda}_k}}\, \sigma_{\boldsymbol{\tau}} \right) \, (\xi_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}} - \xi_{i,\boldsymbol{\lambda}_k,\boldsymbol{\tau}})\; \beta_{i,\boldsymbol{\tau}} \right| \\
&\leq \sigma_{\boldsymbol{\tau}} \max_{k\,:\,i\in\mathcal{A}_{\boldsymbol{\lambda}_k}} \left| \frac{\mu_{\boldsymbol{\lambda}_k} - r_f}{\sigma_{\boldsymbol{\lambda}_k}} \right| \max_{k\,:\,i\in\mathcal{A}_{\boldsymbol{\lambda}_k}} \left| \left(\xi_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}} - \xi_{\boldsymbol{i},\boldsymbol{\lambda}_k,\boldsymbol{\tau}} \right)\; \beta_{i,\boldsymbol{\tau}} \right| \\
&\leq \sigma_{\boldsymbol{\tau}} \left| \frac{\mu_{\boldsymbol{\lambda}^\star} - r_f}{\sigma_{\boldsymbol{\lambda}^\star}} \right| \beta_{i,\boldsymbol{\tau}} \max_{k\,:\,i\in\mathcal{A}_{\boldsymbol{\lambda}_k}} |\xi_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}} - \xi_{i,\boldsymbol{\lambda}_k,\boldsymbol{\tau}}| \\
&= |\mu_{\boldsymbol{\lambda}^\star} - r_f|\, \frac{\sigma_{\boldsymbol{\tau}}}{\sigma_{\boldsymbol{\lambda}^\star}}\, \beta_{i,\boldsymbol{\tau}} \max_{k\,:\,i\in\mathcal{A}_{\boldsymbol{\lambda}_k}} |\xi_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}} - \xi_{i,\boldsymbol{\lambda}_k,\boldsymbol{\tau}}|\,.
\end{aligned}
$$

The last two inequalities use $\mu_{\boldsymbol{\lambda}_k} - \theta_{\boldsymbol{\lambda}_k} \leq \mu_{\boldsymbol{\lambda}_k} - r_f$ and $\max_{k\in\mathcal{K}} \left| \frac{\mu_{\boldsymbol{\lambda}_k} - r_f}{\sigma_{\boldsymbol{\lambda}_k}} \right| \leq \left| \frac{\mu_{\boldsymbol{\lambda}^\star} - r_f}{\sigma_{\boldsymbol{\lambda}^\star}} \right|$, respectively. □

E Proof of Theorem 5

Inspection of the proof of Theorem 2 (see Equation (25)) shows that

$$
|\xi_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}} - \xi_{i,\boldsymbol{\lambda}_k,\boldsymbol{\tau}}| = \left| \sum_{l=2}^{L} \left(\frac{\rho_{i,l}}{\rho_{i,\boldsymbol{\tau}}} \right) \rho_{\boldsymbol{\lambda}_k,l} + \lambda_{i,k} \frac{\sigma^2(\epsilon_i)}{\sigma_i\, \sigma_{\boldsymbol{\lambda}_k}} \right| \leq \left| \sum_{l=2}^{L} \left(\frac{\rho_{i,l}}{\rho_{i,\boldsymbol{\tau}}} \right) \rho_{\boldsymbol{\lambda}_k,l} \right| + \lambda_{i,k} \frac{\sigma^2(\epsilon_i)}{\sigma_i\, \sigma_{\boldsymbol{\lambda}_k}}.
$$

We have

$$
\sum_{l=2}^{L} \left(\frac{\rho_{i,l}}{\rho_{i,\boldsymbol{\tau}}} \right) \rho_{\boldsymbol{\lambda}_k,l} = \sum_{l=2}^{L} \left(\frac{r_{i,l}}{r_{i,\boldsymbol{\tau}}} \right) \rho_{\boldsymbol{\lambda}_k,l} = \sum_{l=2}^{L} m_{i,l}\, \rho_{\boldsymbol{\lambda}_k,l} = f(m_{i,2}, \ldots, m_{i,L})
$$

where $m_{i,l} = r_{i,l}/r_{i,\boldsymbol{\tau}}$ for $l = 2, \ldots, L$. Using $r_{i,\boldsymbol{\tau}}^2 + \sum_{l=2}^{L} r_{i,l}^2 = 1$ we obtain $\sum_{l=2}^{L} m_{i,l}^2 = 1/r_{i,\boldsymbol{\tau}}^2 - 1$.

We now maximize $f(m_{i,2}, \ldots, m_{i,L})$ over $m_{i,2}, \ldots, m_{i,L}$ under the constraint $\sum_{l=2}^{L} m_{i,l}^2 = 1/r_{i,\boldsymbol{\tau}}^2 - 1$. The Lagrange function is

$$
L(m_{2,l}, \ldots, m_{i,L}, \eta) = \sum_{l=2}^{L} m_{i,l}\, \rho_{\boldsymbol{\lambda}_k,l} - \eta \left(\sum_{l=2}^{L} m_{i,l}^2 - 1/r_{i,\boldsymbol{\tau}}^2 + 1 \right)
$$

and the first order conditions are

$$
\begin{aligned}
&(29) \qquad & \rho_{\boldsymbol{\lambda}_l,f_l} - 2\,\eta\, m_{i,l} &= 0 \\
&(30) \qquad & \sum_{l=2}^{L} m_{i,l}^2 - 1/r_{i,\boldsymbol{\tau}}^2 + 1 &= 0.
\end{aligned}
$$

Equation (29) implies

$$m^\star_{i,l} = \frac{\rho_{\boldsymbol{\lambda}_l,l}}{2\,\eta}.$$

We insert this latter expression in (30) and obtain

$$\frac{1}{2\,\eta} = \pm\,\frac{\sqrt{\frac{1}{r^2_{i,\boldsymbol{\tau}}} - 1}}{\sqrt{\sum_{l=2}^{L} \rho^2_{\boldsymbol{\lambda}_k,l}}}$$

and thus

$$m^\star_{i,l} = \pm\,\frac{\sqrt{\frac{1}{r^2_{i,\boldsymbol{\tau}}} - 1}}{\sqrt{\sum_{l=2}^{L} \rho^2_{\boldsymbol{\lambda}_k,l}}}\,\rho_{\boldsymbol{\lambda}_k,l}.$$

It follows

$$\left|\sum_{l=2}^{L} m_{i,l}\,\rho_{\boldsymbol{\lambda}_k,l}\right| \le \sum_{l=2}^{L} \left|m^\star_{i,l}\right|\,\rho_{\boldsymbol{\lambda}_k,l} = \sqrt{\frac{1}{r^2_{i,\boldsymbol{\tau}}} - 1}\,\sqrt{\sum_{l=2}^{L} \rho^2_{\boldsymbol{\lambda}_k,l}}.$$

Therefore

$$|\xi_{\boldsymbol{\lambda}_k,\boldsymbol{\tau}} - \xi_{i,\boldsymbol{\lambda}_k,\boldsymbol{\tau}}| \le \sqrt{\frac{1}{r^2_{i,\boldsymbol{\tau}}} - 1}\,\sqrt{\sum_{l=2}^{L} \rho^2_{\boldsymbol{\lambda}_k,l}} + \lambda_{i,k}\,\frac{\sigma^2(\epsilon_i)}{\sigma_i\,\sigma_{\boldsymbol{\lambda}_k}}.$$

□

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Table I: Numerical example. We use monthly total stock returns from the 2009 edition of the Center for Research in Security Prices (CRSP) file and the one-month US T-bill rate from Ibbotson and Associates. Following the convention, our analysis focuses on ordinary common U.S. stocks listed on the NYSE, AMEX and Nasdaq markets, excluding ADRs, REITs, closed-end-funds, units of beneficial interest, and foreign stocks. We require stocks to have return observations available for the past 60 months in order to estimate their market betas. If price information is no longer available, then a stock is excluded from the analysis and the delisting return or partial monthly return provided by CRSP is used as the last observation. At the end of each month, all stocks that fulfill our data requirements are divided in ten deciles based on their market betas in the past 60 months. For each decile, portfolio returns are computed as the value-weighted average of the returns to the individual stocks. The table includes descriptive statistics for the returns for the sample period from January 1931 to December 2009. The table also shows how the historical average returns can be modified to be consistent with an equilibrium model with one conservative investor ($\zeta_1 = 40$) and one adventurous investor ($\zeta_2 = 400$) who are restricted to make combinations of ten beta deciles and a risk-free Treasury bill without riskless borrowing and short selling. Both investors have the same wealth level ($w_1 = w_2 = 0.5$). The solution is found by minimizing the average absolute deviation from the historical average returns subject to the condition that portfolios P_1 and P_2 (i) are optimal for the the conservative and adventurous investor, respectively, and (ii) add up to the market portfolio, or the value-weighted average of the ten deciles. The table shows the optimal perturbations, adjusted means, and the portfolio weights ($\boldsymbol{\lambda}_k$) and shadow prices ($\boldsymbol{\alpha}_{\boldsymbol{\lambda}_k}$) of the beta deciles in the two optimal portfolios.

	Descriptives					Weights ($\boldsymbol{\lambda}_k$)			Shadow prices ($\boldsymbol{\alpha}_{\boldsymbol{\lambda}_k}$)	
Portfolio	Mean	St. dev.	Mkt beta	Perturb.	Adj. Mean	Mkt	P_1	P_2	P_1	P_2
Low	0.807	4.056	0.572	0.000	0.807	0.118	0.235	0	0	0.147
2	0.868	4.419	0.697	0.012	0.880	0.135	0.270	0	0	0.092
3	1.038	5.201	0.853	-0.068	0.970	0.140	0.280	0	0	0.025
4	0.999	5.610	0.932	0.009	1.008	0.124	0.215	0.034	0	0
5	1.087	6.294	1.056	-0.058	1.029	0.107	0	0.213	0.034	0
6	0.982	6.272	1.135	0.059	1.041	0.101	0	0.202	0.061	0
7	1.056	7.350	1.226	0.000	1.056	0.098	0	0.195	0.091	0
8	1.074	7.991	1.328	-0.003	1.071	0.081	0	0.162	0.131	0
9	1.072	9.053	1.463	0.021	1.093	0.061	0	0.123	0.169	0
High	1.206	10.665	1.670	-0.082	1.124	0.035	0	0.071	0.258	0
T-bill	0.302	0	0	0	0.302	0	0	0	0.082	0.580
Mkt	0.949	5.378	1.000		0.986					
P_1 ($\zeta_1 = 40$)	0.929	4.564	0.761		0.916					
P_2 ($\zeta_2 = 400$)	1.061	7.266	1.239		1.057					

Table II: Full-sample regression results for size-beta stock portfolios. We compute cross-sectional Fama and MacBeth (1973) regressions of monthly excess returns on stock characteristics for 100 size-beta stock portfolios. Regressions are run each month from January 1933 to December 2010 (936 months). The stock characteristics include past 60-month market beta ($\hat{\beta}$), centered beta squared ($(\hat{\beta}-\bar{\beta})^2$), past 60-month (standardized) residual risk ($\hat{\sigma}'(\epsilon)$), log market capitalization of equity (ME), log book-to-market ratio (BtM) and log illiquidity ($Illiq$). One-month lagged one-month return (R_1), one-month lagged 11-month return (R_{12-2}) and 12-month lagged 48-month return (R_{60-13}) equal the log cumulative past returns. The reported coefficients are time-series averages of the monthly regression slopes ($\bar{\gamma}$). The Newey-West corrected t-statistics of these averages are shown in brackets ($t(\bar{\gamma})$). The two bottom rows show the sample mean ($\bar{x}$) and standard deviation ($s(x)$) of the regressors.

		const	$\hat{\beta}$	$(\hat{\beta}-\bar{\beta})^2$	$\hat{\sigma}'(\epsilon)$	ME	BtM	R_1	R_{12-2}	R_{60-13}	$Illiq$	Adj.R^2	ABP
1933M01-2010M12	$\bar{\gamma}$	0.601	0.286									0.133	0.286
	$t(\bar{\gamma})$	3.96	1.49										1.49
	$\bar{\gamma}$	0.749	0.105			-0.108						0.185	0.105
	$t(\bar{\gamma})$	4.27	0.59			-3.08							0.59
	$\bar{\gamma}$	0.803	0.079			-0.090	0.130					0.192	0.079
	$t(\bar{\gamma})$	4.55	0.46			-2.59	2.04						0.46
	$\bar{\gamma}$	0.772	0.089			-0.087	0.108	-3.252	1.346	-0.316		0.242	0.089
	$t(\bar{\gamma})$	4.83	0.57			-2.83	1.77	-4.59	7.39	-4.40			0.57
	$\bar{\gamma}$	0.934	0.077			-0.139	0.131	-3.273	1.404	-0.280	-0.042	0.251	0.077
	$t(\bar{\gamma})$	5.03	0.50			-3.79	2.10	-4.63	7.39	-3.57	-1.29		0.50
1933M01-2010M12	$\bar{\gamma}$	0.606	0.337	-0.492								0.147	0.949
	$t(\bar{\gamma})$	3.88	1.78	-3.32									3.54
	$\bar{\gamma}$	0.751	0.156	-0.471		-0.105						0.199	0.751
	$t(\bar{\gamma})$	4.24	0.88	-3.27		-3.04							2.86
	$\bar{\gamma}$	0.799	0.136	-0.488		-0.086	0.151					0.205	0.744
	$t(\bar{\gamma})$	4.48	0.79	-3.30		-2.47	2.29						2.85
	$\bar{\gamma}$	0.801	0.100	-0.421		-0.085	0.128	-3.387	1.326	-0.329		0.254	0.615
	$t(\bar{\gamma})$	4.87	0.65	-2.98		-2.74	2.09	-4.91	7.28	-4.61			2.56
	$\bar{\gamma}$	0.912	0.092	-0.426		-0.128	0.154	-3.400	1.380	-0.292	-0.029	0.262	0.607
	$t(\bar{\gamma})$	4.86	0.60	-3.09		-3.50	2.43	-4.96	7.26	-3.74	-0.91		2.56
1933M01-2010M12	$\bar{\gamma}$	0.570	0.096		0.189							0.180	0.096
	$t(\bar{\gamma})$	3.57	0.56		1.98								0.56
	$\bar{\gamma}$	0.588	0.110	-0.555	0.228							0.194	0.862
	$t(\bar{\gamma})$	3.57	0.63	-3.50	2.13								3.36
	$\bar{\gamma}$	0.915	0.220	-0.481	-0.047	-0.124						0.218	0.860
	$t(\bar{\gamma})$	4.52	1.28	-3.15	-0.51	-3.95							3.35
	$\bar{\gamma}$	0.979	0.212	-0.493	-0.068	-0.107	0.160					0.225	0.853
	$t(\bar{\gamma})$	4.82	1.29	-3.19	-0.75	-3.41	2.43						3.34
	$\bar{\gamma}$	1.011	0.176	-0.441	-0.061	-0.105	0.151	-3.713	1.419	-0.298		0.268	0.752
	$t(\bar{\gamma})$	5.63	1.16	-2.99	-0.65	-3.98	2.50	-5.32	7.45	-3.91			3.18
	$\bar{\gamma}$	1.016	0.155	-0.466	-0.040	-0.117	0.150	-3.679	1.398	-0.293	-0.010	0.273	0.751
	$t(\bar{\gamma})$	5.08	1.04	-3.27	-0.42	-3.52	2.42	-5.34	7.11	-3.77	-0.30		3.28
	$\bar{\gamma}$	1.046	0.164		-0.086	-0.133	0.128	-3.513	1.444	-0.276	-0.022	0.264	0.164
	$t(\bar{\gamma})$	5.21	1.12		-1.04	-3.70	2.10	-4.94	7.45	-3.52	-0.62		1.12
	$\bar{x}$		1.167	0.137	1.952	0.635	-0.153	0.016	0.169	0.682	3.027		
	$s(x)$		0.406	0.257	0.839	2.686	0.691	0.075	0.265	0.614	3.044		

Table III: Robustness of regression results for size-beta stock portfolios. We compute cross-sectional Fama and MacBeth (1973) regressions of monthly excess returns on stock characteristics for size-beta stock portfolios. Regressions are run each month from January 1933 to December 2010 (936 months). The stock characteristics include past 60-month market beta ($\hat{\beta}$), centered beta squared ($(\hat{\beta}-\bar{\beta})^2$), past 60-month (standardized) residual risk ($\hat{\sigma}'(\epsilon)$), log market capitalization of equity (ME), log book-to-market ratio (BtM) and log illiquidity ($Illiq$). One-month lagged one-month return (R_1), one-month lagged 11-month return (R_{12-2}) and 12-month lagged 48-month return (R_{60-13}) equal the log cumulative past returns. The reported coefficients are time-series averages of the monthly regression slopes ($\bar{\gamma}$). The Newey-West corrected t-statistics of these averages are shown in brackets ($t(\bar{\gamma})$). The first panel makes use of 100 portfolios, including the micro-cap segment, and analyzes the post-1963 period (570 months). The last two panels exclude the micro-cap segment and make use of 80 size-beta stock portfolios. The center panel focuses on the full sample of 936 months, whereas the last panel includes only the post-1963 period. The two bottom rows in each panel show the sample mean ($\bar{x}$) and standard deviation ($s(x)$) of the regressors.

		const	$\hat{\beta}$	$(\hat{\beta}-\bar{\beta})^2$	$\hat{\sigma}'(\epsilon)$	ME	BtM	R_1	R_{12-2}	R_{60-13}	$Illiq$	Adj.R^2	ABP
1963M07-2010M12	$\bar{\gamma}$	0.264	0.164	-0.482	0.153							0.210	0.777
	$t(\bar{\gamma})$	1.15	0.81	-2.31	1.55								2.48
	$\bar{\gamma}$	0.851	0.303	-0.452	-0.110	-0.101	0.160					0.244	0.870
	$t(\bar{\gamma})$	2.76	1.71	-2.16	-1.37	-2.41	1.81						2.71
	$\bar{\gamma}$	0.887	0.288	-0.432	-0.168	-0.109	0.150	-2.145	1.401	-0.197		0.285	0.837
	$t(\bar{\gamma})$	3.26	1.80	-2.13	-1.95	-3.02	1.83	-2.81	5.72	-2.36			2.81
	$\bar{\gamma}$	0.948	0.237	-0.493	-0.119	-0.133	0.152	-2.184	1.382	-0.191	-0.029	0.292	0.853
	$t(\bar{\gamma})$	3.53	1.53	-2.51	-1.32	-3.12	1.79	-2.91	5.34	-2.10	-0.73		2.94
	$\bar{\gamma}$	0.936	0.234		-0.146	-0.130	0.114	-1.970	1.402	-0.196	-0.027	0.281	0.234
	$t(\bar{\gamma})$	3.55	1.47		-1.59	-3.18	1.36	-2.56	5.37	-2.13	-0.66		1.47
	$\bar{x}$		1.116	0.117	2.113	2.181	-0.416	0.016	0.173	0.754	1.435		
	$s(x)$		0.359	0.232	0.859	1.807	0.519	0.064	0.229	0.496	2.466		
1933M01-2010M12 Ex-micro	$\bar{\gamma}$	0.392	0.161	-0.351	0.262							0.198	0.585
	$t(\bar{\gamma})$	2.33	0.97	-2.20	2.77								2.32
	$\bar{\gamma}$	0.838	0.212	-0.379	-0.033	-0.085	0.163					0.229	0.656
	$t(\bar{\gamma})$	3.61	1.44	-2.43	-0.36	-2.62	2.32						2.66
	$\bar{\gamma}$	0.875	0.166	-0.371	-0.074	-0.093	0.163	-3.659	1.470	-0.213		0.275	0.610
	$t(\bar{\gamma})$	4.38	1.25	-2.44	-0.92	-3.33	2.48	-5.73	6.85	-2.88			2.63
	$\bar{\gamma}$	0.898	0.167	-0.374	-0.076	-0.097	0.146	-3.544	1.443	-0.212	-0.011	0.279	0.614
	$t(\bar{\gamma})$	4.27	1.26	-2.49	-0.94	-2.69	2.20	-5.64	6.52	-2.72	-0.31		2.64
	$\bar{\gamma}$	0.998	0.180		-0.142	-0.121	0.110	-3.412	1.466	-0.203	-0.027	0.268	0.180
	$t(\bar{\gamma})$	4.69	1.37		-1.75	-3.32	1.66	-5.36	6.72	-2.58	-0.71		1.37
	$\bar{x}$		1.126	0.123	1.736	1.018	-0.225	0.015	0.157	0.630	2.442		
	$s(x)$		0.374	0.222	0.634	2.515	0.625	0.069	0.244	0.553	2.909		
1963M07-2010M12 Ex-micro	$\bar{\gamma}$	0.114	0.102	-0.502	0.278							0.204	0.701
	$t(\bar{\gamma})$	0.47	0.48	-2.17	2.31								2.09
	$\bar{\gamma}$	0.695	0.220	-0.533	-0.011	-0.082	0.121					0.238	0.848
	$t(\bar{\gamma})$	1.93	1.25	-2.31	-0.10	-1.73	1.25						2.56
	$\bar{\gamma}$	0.811	0.209	-0.568	-0.104	-0.093	0.124	-2.375	1.485	-0.208		0.285	0.893
	$t(\bar{\gamma})$	2.65	1.31	-2.65	-1.05	-2.31	1.40	-2.86	5.32	-2.27			2.98
	$\bar{\gamma}$	0.748	0.218	-0.586	-0.106	-0.074	0.108	-2.222	1.441	-0.217	0.009	0.290	0.920
	$t(\bar{\gamma})$	2.51	1.37	-2.79	-1.04	-1.50	1.20	-2.72	4.97	-2.24	0.17		3.04
	$\bar{\gamma}$	0.839	0.227		-0.167	-0.095	0.046	-2.070	1.441	-0.218	-0.004	0.278	0.227
	$t(\bar{\gamma})$	2.77	1.40		-1.56	-1.93	0.52	-2.43	5.00	-2.18	-0.08		1.40
	$\bar{x}$		1.087	0.112	1.901	2.489	-0.464	0.015	0.156	0.681	0.788		
	$s(x)$		0.348	0.234	0.653	1.725	0.495	0.061	0.213	0.440	2.234		

Table IV: Full-sample regression results for individual stocks. We compute cross-sectional Fama and MacBeth (1973) regressions of monthly excess returns on stock characteristics for individual stocks. Regressions are run each month from January 1933 to December 2010 (936 months). The stock characteristics include past 60-month market beta ($\hat{\beta}$), centered beta squared ($(\hat{\beta}-\bar{\beta})^2$), past 60-month (standardized) residual risk ($\hat{\sigma}'(\epsilon)$), log market capitalization of equity (ME), log book-to-market ratio (BtM) and log illiquidity ($Illiq$). One-month lagged one-month return (R_1), one-month lagged 11-month return (R_{12-2}) and 12-month lagged 48-month return (R_{60-13}) equal the log cumulative past returns. The market beta and past return variables are windsorized at first and 99th percentile; whereas size, book-to-market, idiosyncratic risk and illiquidity are windsorized at the 99th percentile values. The reported coefficients are time-series averages of the monthly regression slopes ($\bar{\gamma}$). The Newey-West corrected t-statistics of these averages are shown in brackets ($t(\bar{\gamma})$). The two bottom rows show the sample mean ($\bar{x}$) and standard deviation ($s(x)$) of the regressors.

		const	$\hat{\beta}$	$(\hat{\beta}-\bar{\beta})^2$	$\hat{\sigma}'(\epsilon)$	ME	BtM	R_1	R_{12-2}	R_{60-13}	$Illiq$	Adj.R^2	ABP
1933M01 - 2010M12	$\bar{\gamma}$	0.898	0.097									0.031	0.097
	$t(\bar{\gamma})$	5.74	0.76										0.76
	$\bar{\gamma}$	0.486	0.018			-0.122						0.050	0.018
	$t(\bar{\gamma})$	3.27	0.16			-2.69							0.16
	$\bar{\gamma}$	0.626	0.045			-0.092	0.243					0.057	0.045
	$t(\bar{\gamma})$	4.24	0.43			-2.15	4.71						0.43
	$\bar{\gamma}$	0.610	0.053			-0.068	0.189	-8.258	0.940	-0.171		0.088	0.053
	$t(\bar{\gamma})$	4.34	0.49			-2.10	3.59	-17.16	6.27	-2.18			0.49
	$\bar{\gamma}$	0.477	0.047			-0.108	0.193	-8.271	0.958	-0.166	-0.028	0.093	0.047
	$t(\bar{\gamma})$	2.08	0.45			-2.44	3.76	-16.98	6.32	-2.17	-0.88		0.45
1933M01 - 2010M12	$\bar{\gamma}$	0.914	0.146	-0.241								0.034	0.442
	$t(\bar{\gamma})$	5.83	1.14	-4.32									2.71
	$\bar{\gamma}$	0.482	0.076	-0.297		-0.131						0.052	0.455
	$t(\bar{\gamma})$	3.27	0.67	-5.14		-2.88							2.82
	$\bar{\gamma}$	0.619	0.097	-0.271		-0.100	0.235					0.059	0.440
	$t(\bar{\gamma})$	4.23	0.92	-4.44		-2.32	4.61						2.70
	$\bar{\gamma}$	0.609	0.093	-0.236		-0.073	0.187	-8.282	0.938	-0.165		0.090	0.397
	$t(\bar{\gamma})$	4.35	0.85	-3.84		-2.24	3.59	-17.28	6.23	-2.13			2.45
	$\bar{\gamma}$	0.502	0.091	-0.238		-0.108	0.192	-8.306	0.955	-0.158	-0.024	0.095	0.401
	$t(\bar{\gamma})$	2.16	0.85	-3.76		-2.45	3.80	-17.09	6.26	-2.11	-0.76		2.42
1933M01 - 2010M12	$\bar{\gamma}$	1.010	0.103		0.027							0.049	0.103
	$t(\bar{\gamma})$	7.10	1.14		0.32								1.14
	$\bar{\gamma}$	1.008	0.132	-0.256	0.050							0.051	0.476
	$t(\bar{\gamma})$	7.10	1.44	-4.18	0.55								3.08
	$\bar{\gamma}$	0.647	0.225	-0.255	-0.178	-0.194						0.061	0.556
	$t(\bar{\gamma})$	4.28	2.33	-4.37	-3.42	-4.98							3.60
	$\bar{\gamma}$	0.722	0.208	-0.250	-0.157	-0.161	0.176					0.066	0.529
	$t(\bar{\gamma})$	4.72	2.25	-4.10	-3.12	-4.21	3.75						3.37
	$\bar{\gamma}$	0.722	0.191	-0.212	-0.145	-0.125	0.131	-8.386	0.957	-0.177		0.097	0.473
	$t(\bar{\gamma})$	5.01	1.98	-3.37	-2.59	-4.43	2.65	-17.20	6.42	-2.29			3.01
	$\bar{\gamma}$	0.626	0.203	-0.211	-0.167	-0.164	0.137	-8.400	0.979	-0.171	-0.024	0.100	0.486
	$t(\bar{\gamma})$	2.74	2.12	-3.29	-3.06	-4.29	2.82	-16.97	6.50	-2.28	-0.79		3.02
	$\bar{\gamma}$	0.612	0.172		-0.182	-0.166	0.135	-8.363	0.989	-0.179	-0.026	0.098	0.172
	$t(\bar{\gamma})$	2.7	1.88		-3.41	-4.32	2.77	-16.89	6.66	-2.33	-0.84		1.88
	$\bar{x}$		1.117	0.493	2.934	-2.590	-0.392	0.002	0.027	0.251	-2.463		
	$s(x)$		0.716	0.987	1.874	2.146	0.973	0.142	0.480	0.867	2.622		

Table V: Robustness of regression results for individual stocks. We compute cross-sectional Fama and MacBeth (1973) regressions of monthly excess returns on stock characteristics for individual stocks. Regressions are run each month from January 1933 to December 2010 (936 months). The stock characteristics include past 60-month market beta ($\hat{\beta}$), centered beta squared ($(\hat{\beta}-\bar{\beta})^2$), past 60-month (standardized) residual risk ($\hat{\sigma}'(\epsilon)$), log market capitalization of equity (ME), log book-to-market ratio (BtM) and log illiquidity ($Illiq$). One-month lagged one-month return (R_1), one-month lagged 11-month return (R_{12-2}) and 12-month lagged 48-month return (R_{60-13}) equal the log cumulative past returns. The reported coefficients are time-series averages of the monthly regression slopes ($\bar{\gamma}$). The Newey-West corrected t-statistics of these averages are shown in brackets ($t(\bar{\gamma})$). The first panel makes use of the entire cross-section of stocks, including the micro-cap segment, and analyzes the post-1963 period (570 months). The last two panels exclude the micro-cap segment. The center panel focuses on the full sample of 936 months, whereas the last panel includes only the post-1963 period. The two bottom rows in each panel show the sample mean ($\bar{x}$) and standard deviation ($s(x)$) of the regressors.

		const	$\hat{\beta}$	$(\hat{\beta}-\bar{\beta})^2$	$\hat{\sigma}'(\epsilon)$	ME	BtM	R_1	R_{12-2}	R_{60-13}	$Illiq$	Adj.R^2	ABP
1963M07 - 2010M12	$\bar{\gamma}$	0.949	0.121	-0.170	-0.101							0.039	0.343
	$t(\bar{\gamma})$	4.80	1.21	-3.26	-1.53								2.51
	$\bar{\gamma}$	0.732	0.269	-0.205	-0.186	-0.139	0.189					0.051	0.534
	$t(\bar{\gamma})$	3.58	2.70	-3.40	-3.10	-3.89	3.68						3.57
	$\bar{\gamma}$	0.650	0.269	-0.149	-0.177	-0.107	0.148	-6.901	0.984	-0.154		0.076	0.470
	$t(\bar{\gamma})$	3.23	2.52	-2.30	-2.87	-3.44	2.56	-12.88	5.91	-2.16			2.93
	$\bar{\gamma}$	0.189	0.249	-0.132	-0.190	-0.210	0.153	-6.855	1.029	-0.160	-0.079	0.080	0.428
	$t(\bar{\gamma})$	0.62	2.37	-1.99	-3.20	-3.92	2.72	-12.73	6.04	-2.23	-1.88		2.60
	$\bar{\gamma}$	0.190	0.225		-0.194	-0.211	0.152	-6.825	1.031	-0.162	-0.080	0.078	0.225
	$t(\bar{\gamma})$	0.63	2.21		-3.28	-3.94	2.69	-12.70	6.04	-2.25	-1.92		2.21
	$\bar{x}$		1.107	0.527	3.106	-2.434	-0.469	0.001	0.017	0.234	-2.612		
	$s(x)$		0.736	1.039	1.915	2.150	0.952	0.146	0.495	0.878	2.685		
1933M01 - 2010M12, Ex Micro	$\bar{\gamma}$	0.897	0.199	-0.240	-0.005							0.060	0.500
	$t(\bar{\gamma})$	6.39	1.80	-4.13	-0.07								3.17
	$\bar{\gamma}$	0.752	0.204	-0.241	-0.119	-0.133	0.120					0.078	0.509
	$t(\bar{\gamma})$	4.74	1.87	-4.14	-2.06	-3.74	2.35						3.19
	$\bar{\gamma}$	0.678	0.172	-0.209	-0.122	-0.127	0.097	-6.356	1.139	-0.124		0.113	0.445
	$t(\bar{\gamma})$	4.53	1.69	-3.33	-2.04	-4.62	1.84	-13.86	6.54	-1.69			2.89
	$\bar{\gamma}$	0.585	0.182	-0.217	-0.143	-0.162	0.104	-6.354	1.150	-0.119	-0.027	0.116	0.466
	$t(\bar{\gamma})$	2.63	1.79	-3.43	-2.47	-4.11	1.99	-13.71	6.54	-1.66	-0.97		2.99
	$\bar{\gamma}$	0.582	0.156		-0.157	-0.162	0.100	-6.305	1.149	-0.123	-0.028	0.113	0.156
	$t(\bar{\gamma})$	2.62	1.58		-2.72	-4.07	1.91	-13.59	6.58	-1.70	-0.98		1.58
	$\bar{x}$		1.145	0.356	2.122	-1.104	-0.530	0.006	0.088	0.467	-4.093		
	$s(x)$		0.605	0.795	1.202	1.870	0.903	0.114	0.382	0.698	2.208		
1963M07 - 2010M12, Ex Micro	$\bar{\gamma}$	0.802	0.191	-0.216	-0.125							0.052	0.470
	$t(\bar{\gamma})$	3.98	1.50	-3.65	-1.78								2.84
	$\bar{\gamma}$	0.743	0.244	-0.223	-0.156	-0.125	0.119					0.068	0.534
	$t(\bar{\gamma})$	3.30	1.97	-3.58	-2.03	-3.00	1.94						3.15
	$\bar{\gamma}$	0.631	0.215	-0.154	-0.161	-0.122	0.097	-4.880	1.044	-0.116		0.098	0.421
	$t(\bar{\gamma})$	2.91	1.86	-2.13	-2.13	-3.39	1.52	-9.39	4.66	-1.71			2.53
	$\bar{\gamma}$	0.223	0.197	-0.150	-0.174	-0.210	0.104	-4.811	1.073	-0.126	-0.075	0.101	0.396
	$t(\bar{\gamma})$	0.74	1.72	-2.04	-2.42	-3.72	1.66	-9.28	4.72	-1.84	-1.99		2.34
	$\bar{\gamma}$	0.231	0.181		-0.184	-0.210	0.102	-4.760	1.065	-0.126	-0.076	0.098	0.181
	$t(\bar{\gamma})$	0.76	1.59		-2.52	-3.70	1.61	-9.17	4.70	-1.83	-1.99		1.59
	$\bar{x}$		1.143	0.391	2.282	-0.542	-0.668	0.005	0.083	0.482	-4.677		
	$s(x)$		0.630	0.873	1.264	1.536	0.867	0.118	0.395	0.687	1.978		

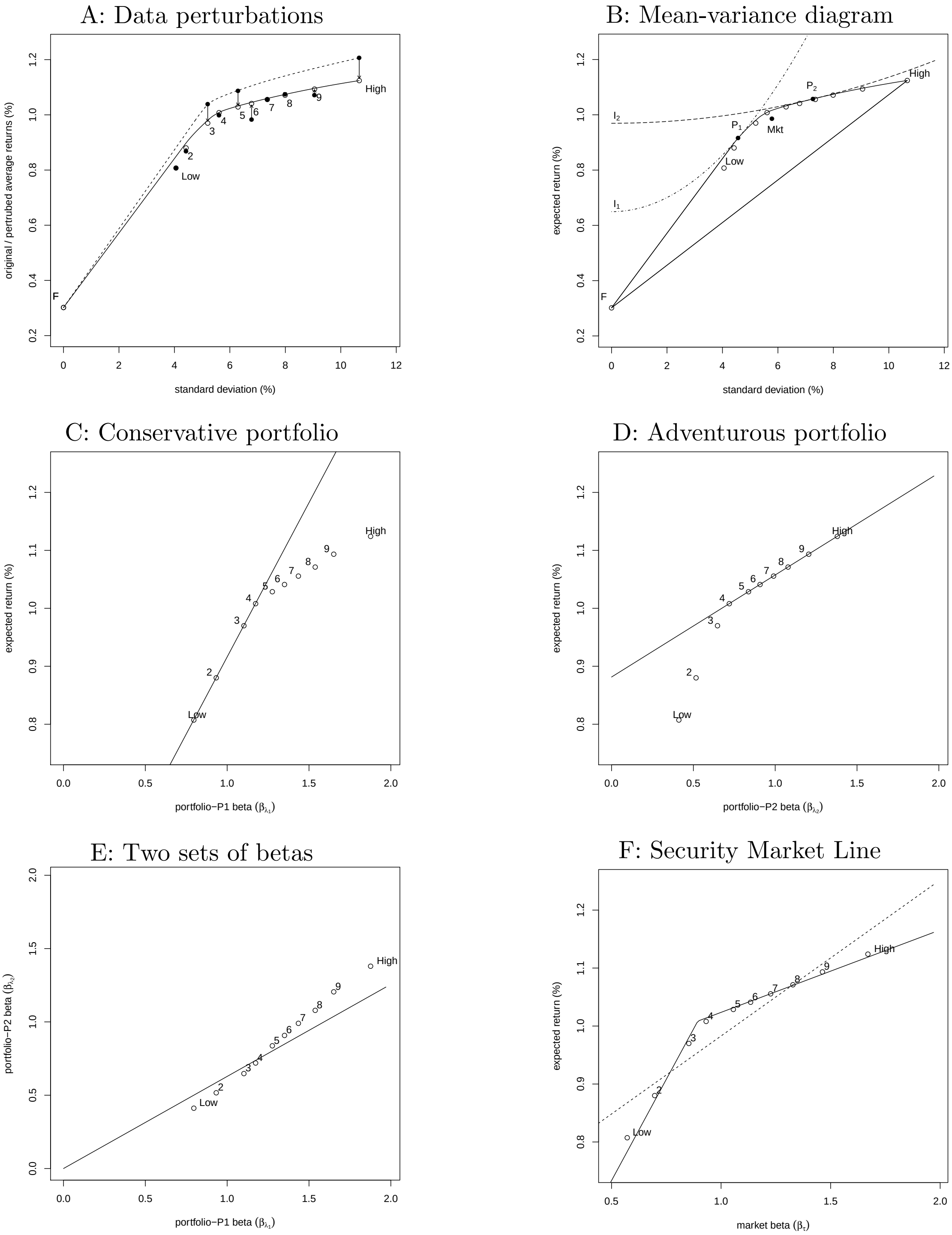

Figure 1: Numerical example. The figure illustrates the data set, method and results described in Table I. Panel A shows a mean-variance diagram with the expected returns and standard deviations of the ten beta deciles and the Treasury bill (open dots), together with the efficient frontier for the case without riskless borrowing and short sales (solid curve). For the sake of comparison, we also show the original, unadjusted average returns (closed dots) and the associated efficient frontier (dashed curve). Panel B shows the full portfolio possibilities set (or all convex combinations of the base assets) and adds the optimal indifference curves (I_1 and I_2) and portfolios (P_1 and P_2) of the two investors, together with the market portfolio (Mkt). Panel C plots the expected returns against the portfolio betas relative to the conservative investor's portfolio (P_1). The solid line represents the portfolio optimality condition; the line connects the conservative investor's active assets and envelops her inactive assets. Panel D shows a similar plot for the adventurous investor and her portfolio P_2. Panel E shows that the betas relative to P_1 are nearly proportional to the betas relative to P_2, reflecting the very high correlation between the two portfolios. Panel F shows the relation between expected return and market beta. The dashed, straight line represents represents the classical SML, or average of the two lines shown in Panel C and D. The solid, kinked line represents our CSML approximation (17), which combines the two sets of optimality conditions under the assumption that P_1 and P_2 have zero non-market factor loadings.