TÜSİAD-KOÇ UNIVERSITY ECONOMIC RESEARCH FORUM
WORKING PAPER SERIES

COMPETITIVE EQUILIBRIA IN DECENTRALIZED MATCHING WITH INCOMPLETE INFORMATION

Alp E. Atakan

Working Paper 1031
October 2010

TÜSİAD-KOÇ UNIVERSITY ECONOMIC RESEARCH FORUM
Rumeli Feneri Yolu 34450 Sarıyer/Istanbul

COMPETITIVE EQUILIBRIA IN DECENTRALIZED MATCHING WITH INCOMPLETE INFORMATION

ALP E. ATAKAN

Abstract.

This paper shows that all perfect Bayesian equilibria of a dynamic matching game with two-sided incomplete information of independent private values variety are asymptotically Walrasian. Buyers purchase a bundle of heterogeneous, indivisible goods and sellers own one unit of an indivisible good. Buyer preferences and endowments as well as seller costs are private information. Agents engage in costly search and meet randomly. The terms of trade are determined through a Bayesian mechanism proposal game. The paper considers a market in steady state. As discounting and the fixed cost of search become small, all trade takes place at a Walrasian price. However, a robust example is presented where the limit price vector is a Walrasian price for an economy where only a strict subsets of the goods in the original economy are traded, i.e, markets are missing at the limit. Nevertheless, there exists a sequence of equilibria that converge to a Walrasian equilibria for the whole economy where all markets are open.

Keywords: Matching and Bargaining, Search, Foundations for Perfect Competition, Two-sided Incomplete Information
JEL Classification Numbers: C73, C78, D83.

1. Introduction

This paper shows that all equilibria of a dynamic matching game with two-sided incomplete information of the independent private values variety converge to Walrasian (or competitive) equilibria, as search frictions disappear. In the model each buyer aims to purchase a bundle of heterogeneous, indivisible objects and each seller owns one unit of a heterogeneous indivisible good (as in Kelso and Crawford (1982) or Gul and Stacchetti (1999)). Buyer preferences and endowments as well as seller costs are private information. Agents engage in costly search and meet randomly in a market that remains in steady state. The terms of trade are determined through a Bayesian mechanism proposal game.

Numerous researchers have explored the non-cooperative foundations for competitive equilibria in indivisible goods markets using dynamic matching games. Previous work has

Date: First version February, 2007. This version July, 2008.

focused almost exclusively on markets for an homogeneous good and has assumed complete information until recently. In particular, Gale (1987) and Mortensen and Wright (2002) establish convergence of dynamic matching game equilibria to competitive equilibria as search friction disappear under complete information, while Satterthwaite and Shneyerov (2007) extend the analysis to the two-sided incomplete information case.

Often cited examples of markets, where indivisible goods are exchanged through bilateral negotiations, are the labor and the housing markets. Although cited as motivating examples, neither of these markets fit the mold of a market for an homogeneous good where buyers only differ in their valuations for the good, and sellers only differ in their cost of providing the good. For example, in the labor market potential employees differ in their productivity and their disutility of labor. Firms usually search for multiple employees, that may complement or substitute each other. Also, the vacancies in the firms are rarely exactly alike, and an employees productivity may depend crucially on the type of vacancy that a firm has available. In the housing market the potential homes are far from being homogenous and buyers in search of homes may have diverse needs. Moreover, many home purchases are bundles that include the home, nearby parking, architectural services for the home and brokerage services for the transaction. This paper presents a dynamic matching game, with two sided incomplete information, that preserves many of the attributes of markets such as the labor market and the housing market.

A brief description of the model presented here is as follows: In each period a unit measure of each type (of buyers and sellers) from a finite set of types is available for entry and those who expect a non-negative return voluntarily enter the market. The market is in steady-state with the measure of agent types endogenously determined to balance the flow of types through the economy. Once in the market, each agent pays a per period cost, and receives a "draw" from the distribution of active players. Also, finding a bargaining partner takes time and agents discount the future. The probability that any buyer (or seller) is paired with a particular type is proportional to the frequency of that type among all sellers (buyers) active in steady state. After two agents are paired, nature designates a proposer, the proposer offers a mechanism, and the responder decides whether to participate (i.e., the Bayesian mechanism proposal game of Maskin and Tirole (1990) is played). During this bargaining stage buyer preferences and endowments, as well as, seller costs are private information. The good that the seller offers, however, is observed by the buyer. If a meeting between a pair results in a trade, then the seller leaves the market, otherwise the agents return to the population of active players. Buyers leave the market voluntarily after they have purchased all the goods that they want.

The competitive equilibrium benchmark under consideration is a "flow" equilibrium as in Gale (1987) or Satterthwaite and Shneyerov (2007), generalized to accommodate heterogenous goods and multi-unit demand. In each period, flow supply is the measure of sellers of a particular good entering the market and flow demand is the measure of agents willing to purchase a particular good entering the market. In a flow equilibrium, the buyer and seller continuation values, which are the implicit prices, equate flow supply to flow demand for each of the goods traded in the market.

The first central result of this paper shows that as the discount factor $\delta \to 1$ and the explicit search costs $c \to 0$, all trade takes place at competitive prices. As search becomes increasingly cheap, buyers wait until they have accumulated their most favored bundle. While accumulating these goods, buyers reject "high" prices. Also, sellers become more discerning and wait until they receive the best price offer possible. At the limit incomplete information stops playing a role, trade in each good occurs at a unique price and each buyer purchases their most preferred bundle at these prices. However, the limit price vector may not comprise a competitive price vector for the whole economy. Instead, the limit price may be a competitive price vector for a economy where only a strict subset of goods are traded. This is because markets for some goods are possibly "closed" (or "missing") at the limit.

The second central result of this paper establishes that a search equilibrium exists for any configuration of search frictions. In this equilibrium all proposers optimally choose take-it-or-leave-it offers (à la Riley and Zeckhauser (1983)) from a rich set of possible mechanisms. Also, when small search frictions are small, the paper shows that an equilibrium exists where the markets for all goods are open. Consequently, there exists a sequence of equilibria that converge to a competitive equilibrium for the whole economy.

Although the literature on dynamic matching and search is vast, Satterthwaite and Shneyerov (2007) is the work most closely related to this one. Satterthwaite and Shneyerov (2007) established that equilibria of a dynamic matching game converge to a competitive equilibrium in the case of a single homogeneous good and two sided incomplete information. Also, in a market for a homogeneous good, Satterthwaite and Shneyerov (2008) show convergence to a competitive equilibrium with an exogenously given exit rate; Lauermann (2008) shows convergence does not depend on the distribution of bargaining power; and Shneyerov and Wong (2007) establish results on the rate of convergence.

The analysis provided here differs from the previous literature in two main respects. First, the homogeneous good, unit demand restrictions are lifted. In a search market for a homogeneous good the limit of any sequence of stable equilibria is competitive and so efficient. In contrast, with heterogeneous goods, there are robust examples where some markets are closed at the limit and the limit equilibrium is inefficient.[1] Second, in all previous work the bargaining protocol, that governs the interaction of buyers and sellers, is exogenously imposed. For example, in Satterthwaite and Shneyerov (2007), Satterthwaite and Shneyerov (2008) and Shneyerov and Wong (2007) the buyers and sellers that meet, are assumed to participate in a double auction where any seller bids her continuation value truthfully. In contrast, here the proposer is allowed to choose any finite mechanism. So, strategic behavior is allowed for both the proposing and responding agents and also a large set of bargaining protocols are permitted. Consequently, the results here show that an asymptotically efficient bargaining protocol will be endogenously chosen by individual agents in equilibrium.

In related models presented in DeFraja and Sakovics (2001), Serrano (2002) and Wolinsky (1990), convergence to a competitive equilibrium fails. The failure of convergence to competitive equilibrium is caused by the bilateral bargaining protocol in Serrano (2002); results from the inefficiency of aggregating common value information through bilateral meeting in Wolinsky (1990); and is due to a "clones" assumption in DeFraja and Sakovics (2001) (see Lauermann (2006) for a detailed discussion of these issues).

The paper proceeds as follows: Section 2 outlines the dynamic matching and bargaining game as well as the competitive benchmark, Section 3.1 presents the main results that show convergence to a competitive equilibrium, Section 3.2 outlines the equilibrium existence argument, and Section 4 concludes. Proofs that are not included in the main text are in the Appendix.

2. The Model

Buyers and sellers in the economy search for possible trading partners over the infinite horizon. Each seller owns one indivisible good for sale and each buyer wants to purchase a bundle of the indivisible goods offered for sale. The game progresses in discrete time and agents discount the future with a common discount factor $\delta = e^{-r\Delta} \in [0, 1]$, where Δ is the period length and r is the discount rate. In each period, an agent incurs a positive explicit search cost $c = \kappa\Delta > 0$ and meets pairwise with a potential partner.[2] Either the buyer or the seller is designated as the proposer and then the pair play a three-stage Bayesian mechanism choice game. The mechanism choice game is exactly the game thoroughly analyzed in Maskin and Tirole (1990). The probability that the buyer is designated as the

[1] With a homogeneous good there is always a no trade equilibrium. However, this equilibrium is not stable. In contrast, with heterogeneous goods Example 1 demonstrates a sequence of stable equilibria with an inefficient limit.

[2] Although, I assume that all agents share a common discount factor δ and explicit search cost, c, this is for convenience only. All results in the paper go through even if agents have heterogeneous search costs.

proposer is $\beta \in (0,1)$. In the first stage, the proposer offers a mechanism chosen from the set of feasible mechanisms. Throughout the paper *assume that a take-it-or-leave-it offer is a feasible mechanism choice for the proposer.* In section 3, the argument for existence further assumes that the set of feasible mechanisms is the set of all finite mechanisms as in Maskin and Tirole (1990). In the second stage, the responder chooses whether to participate in the mechanism. If the responder accepts to participate in the mechanism, then in the third stage the agents play the mechanism and the mechanism chooses the probability with which a trade occurs and specifies the transfers to be paid by the buyer to the seller. Sellers who trade permanently leave the market. Buyers remain in the economy until they have purchased all the goods that they want, and then they leave the market and consume their bundle. Agents who fail to trade return to the searching population. The distribution of agents searching for trading partners is assumed to remain in steady state. Utility is transferable. In particular, if a buyer of type b consumes bundle G, then she enjoys utility h_{bG}. A seller incurs cost r_s when she sells her good. So trade between b and sellers $s \in G$ creates total transferable utility $f_{bG} = h_{bG} - \sum_{s\in G} r_s$.

2.1. **Population of Types and Private Information.** Let I and S denote the finite sets of initial buyer and seller types. A seller's type specifies the good she owns, x_s, and her reservation value (or cost) r_s. A buyer's initial type specifies the buyer's utility function $h_i : \mathcal{P}(S) \to \mathbb{R}$, where $\mathcal{P}(S)$ denotes the set of all subsets of S. The utility function satisfies:

(i) Normalization: $h_{i\emptyset} = 0$,
(ii) Monotonicity: If $G \supset A$, then $h_{iG} \geq h_{iA}$,
(iii) Identity Independence: For any s and s' with $x_s = x_{s'}$ (i.e., for sellers s and s' who own the same good), $h_{iG\cup\{s\}} = h_{iG\cup\{s'\}}$ for all G.

Once in the market, a buyer's type changes after each trade and includes information on all trades that the agent has made, and consequently, the goods that the buyer owns. So, refer to a buyer type by $b = i_G$, where $i \in I$ is the initial type, i.e., her utility function, and G is the set of seller types with whom she has already traded. Consequently, the set of buyer types, potentially available for trade in the market, is $B = I \times \mathcal{P}(S)$. For $b \in B$, $G(b) \subset S$ denotes the sellers with whom b has already traded; and $i(b) \in I$ denotes the initial type of b. The notation $b \cup s$ denotes a type b' with $i(b) = i(b')$ and $G(b') = G(b) \cup \{s\}$. Similarly, the notation $b \setminus s$ denotes a type b' with $i(b) = i(b')$ and $G(b') = G(b) \setminus s$.

In each period, a unit measure of each $i \in I$ and $s \in S$ are available to enter the market. Consequently, in each period a measure $|I|$ of buyers and measure $|S|$ of sellers potentially enter the market.[3] Buyers and sellers, who do not enter the market in a given period, are

[3]The assumption that there is unit entry of each type is without loss of generality and is only for expositional convenience.

assumed to have opted for an outside option and are thus not available for entry in any subsequent periods.

Let $l = (l_1, ..., l_b, ..., l_{|B|}, l_1, ..., l_{|S|})$ denote the steady state measure of buyers and sellers present in the market. The steady state probability for any seller of meeting buyer b, or any buyer meeting a seller s in a given period is

$$p_b = \frac{l_b}{\max\{L_B, L_S\}} \quad \text{or} \quad p_s = \frac{l_s}{\max\{L_B, L_S\}} \tag{1}$$

where $L_B = \sum_{b \in B} l_b$ and $L_S = \sum_{s \in S} l_s$. Assume that *the agents know the distribution of types in the economy.*[4] In particular, the (sub) probability measures p_b and p_s (or type distributions) are commonly known by all agents. *Observe that the population of types available for entry in each period is a primitive of the model and is given exogenously. In contrast, the steady state measure, l, is determined endogenously by the measure of agents entering and exiting the market in each period.*

The analysis here assumes *independent private values.* More precisely, if a buyer and seller consummate a trade, then the payoff to each agent depends on the terms of trade and the agent's own private information; but does not depend on the trading partner's private information, i.e., there is no "Lemons" problem. Reference to this assumption, which is stated formally below, is omitted from the statements of the results presented since it is maintained throughout the paper.

Assumption. Independent Private Values. *If buyer $b = i_G$ and seller s meet, then the buyer observes, x_s, the good that seller s has for sale, while the buyer type b and seller cost r_s remain as private information.*

2.2. **Agent Behavior, Strategies and Beliefs.** Let σ_j denote a strategy for type j, π_j denote beliefs for type j, $\sigma = (\sigma_j)_{j \in I}$ denote a strategy profile, and $\pi = (\pi_j)_{j \in I}$ denote a profile of beliefs. The paper focuses on equilibria where all agents use stationary (time-invariant) strategies ($\sigma^t = \sigma$ for all t); beliefs are stationary ($\pi^t = \pi$ for all t); and agents of the same type use the same strategy and entertain the same beliefs. Since agents know the distribution of types in the economy, an agent's belief that she will meet an agent of type j coincides with the actual probability of meeting this agent (i.e., the steady state probability of meeting j, p_j). Consequently, an agent's belief at the start of any period is given by the steady state distribution. At other points of the stage game played during a period, beliefs

[4] This requirement is stronger than what is needed for showing convergence to a competitive equilibrium. As long as the support of any agent's prior belief about the distribution of agents in the economy coincides with the support of the steady state distribution, the convergence results presented in the paper will continue to hold.

are obtained, where possible, using Bayes' rule by conditioning on equilibrium strategies as well as observed characteristics and actions of their opponents for the period.[5]

At the start of each period, a strategy determines whether the agent remains in (or enters) the market and pays the cost c. Denote by $\sigma_j(in)$ the probability that agent j remains in (or enters) the market at the start of any period.

If j is paired in the current period and is the proposer, then the strategy σ_j specifies a mechanism. If agent j is the responder, then the strategy specifies whether she accepts to participate in the mechanism. If the responder accepts to participate, then the agents send messages chosen from the message space specified by the mechanism; and the mechanism chooses a probability of trade and the transfer to be paid by the buyer to the seller.

Given a profile of strategies and beliefs, let the match probability $m_{bs}(\sigma, \pi)$ (or $m_{sb}(\sigma, \pi)$) denote the probability that b and s trade, given that the two are paired in the period and b (or s) is chosen as the proposer. Let $M_{bs} = M_{sb} = \beta m_{bs} + (1 - \beta)m_{sb}$ denote the total probability of trade, given that b and s are paired in the period. Also, let $t_{bs}(\sigma, \pi)$ (or $t_{sb}(\sigma, \pi)$) denote the expected transfer paid by the buyer to the seller, given that b and s are paired in the period and b (or s) is chosen as the proposer.[6] In what follows, match probabilities and transfers will be denoted m_{ij} and t_{ij} with the dependence on σ and π suppressed for notational convenience.

The period reward for a buyer b equals $h_{i(b)G(b)}$ if the buyer chooses to exit the market at the start of the period; equals $-c$ if she chooses to remain in the market but fails to meet a seller; and equals $-c - t_{bs}$ (or $-c - t_{sb}$) if she gets to propose (or respond) to seller s. Likewise, the period reward for a seller s equals zero if the seller chooses to exit the market at the start of the period; equals $-c$ if she chooses to remain in the market but fails to meet a buyer; and equals $-c + t_{sb}$ (or $-c + t_{bs}$) if she gets to propose (or respond) to buyer b. If an agent has exited the market in a prior period, then the period reward for that agent is equal to zero 0. Given that all other agent in the economy behave according to strategies σ and beliefs π, each agent of type j chooses their strategy σ_j to maximize their expected discounted stream of utility.

2.3. **Steady State.** As stated earlier, the measure of agents in the economy is assumed to remain in steady state (i.e., $l_{t+1} = l_t = l$ for all t). The steady state assumption requires that the inflow of type b buyers (or type s sellers) into the market in each period must equal

[5]Observe that all beliefs obtained through Bayes' rule are stationary as a consequence of the steady state assumption on the distribution of types and the assumption that agents use stationary strategies. All other belief, that is beliefs (conditional on zero probability events) that cannot be obtained using Bayes' rule, are further assumed stationary.

[6]For example, if the probability of trade is $m_{bs}(\sigma, \pi)$ and the buyer pays the seller t in case a trade occurs and zero otherwise, then the expected transfer $t_{bs}(\sigma, \pi) = m_{bs}t$.

the outflow of that type exiting the market in each period. Consequently, given strategies and implied match probabilities, the steady state measure must satisfy Equations (2), (3) and (4) given below, in equilibrium. The left hand side of Equation (2) gives the outflow of type s sellers resulting from successful trades or voluntary exit from the market; and the righthand side gives the inflow of new type s sellers. The left side of the Equation (4) (or Equation (3) for types with $b(G) = \emptyset$) gives the outflow of type b buyers resulting from agents leaving the market or transforming into another type following a trade; and the right hand side gives the inflow of type b buyers via new entry or as a result of buyers of another type being transformed into type b following a trade. The steady state equations are as follows:

$$l_s(\sum_{b\in B} p_b M_{bs} + \sigma_s(out)) = \sigma_s(in) \tag{2}$$

$$l_b(\sum_{s\in S} p_s M_{bs} + \sigma_b(out)) = \sigma_b(in) \tag{3}$$

for all types $b \in B$ with $b(G) = \emptyset$ and all sellers $s \in S$, where $\sigma_b(out) = (1-\sum_{s\in S} p_s M_{bs})(1-\sigma_b(in))$ denotes the fraction of type b buyers, who failed to trade in the previous period, that choose to leave at start of the current period; and $\sigma_b(in) \leq 1$ is the flow of new buyers into the market at the start of the period. Also,

$$l_b(\sum_{s\in S} p_s M_{bs} + \sigma_b(out)) = \sigma_b(in) \sum_{s\in G(b)} l_{b\setminus s} p_s M_{b\setminus ss} \tag{4}$$

for $b \in B$ with $b(G) \neq \emptyset$, where $\sigma_b(in) \sum_{s\in G(b)} l_{b\setminus s} p_s M_{b\setminus ss}$ is the measure of newly created type b buyers who remain in the market, that is, buyer types, that were an "s" away from type b, who traded with a type s in the previous period.

2.4. **Search Equilibrium.** A steady state search equilibrium is comprised of a strategy profile σ, a profile of beliefs π and a steady state measure l, that are all mutually compatible. That is to say, the measure l satisfies the steady state equations, given that agents behave according to strategy profile σ and the profile of belief π; and the strategy profile σ and belief π, comprises a perfect Bayesian equilibrium for the three-stage mechanism proposal game, given that types are drawn according to the steady state measure l.

2.5. **Values.** Let v_k denote the expected discounted value for a type k agent given match probabilities m, expected transfers t, and steady state distibution of types p. The expected future value at the start of a period for a buyer equals the maximum of the value of remaining in the market and the value of leaving the market and consuming the bundle that she owns, that is, $v_b = \max\{v_b(in), h_{bG(b)}\}$. The value of remaining in the economy, $v_b(in)$, satisfies the following Bellman equation:

$$v_b(in) = -c + \sum\nolimits_s p_s\beta(m_{bs}\delta v_{b\cup s} - t_{bs}) + \sum\nolimits_s p_s(1-\beta)(m_{sb}\delta v_{b\cup s} - t_{sb}) \\ + (1 - \sum\nolimits_s p_s\beta m_{bs} - \sum\nolimits_s p_s(1-\beta)m_{sb})\delta v_b.$$

In words, buyer b pays the search (sampling) cost c, then successfully makes a trade as the responder with seller s with probability $(1-\beta)p_s m_{sb}$; makes a trade when she proposes to buyer s with probability $\beta p_s m_{bs}$; and does not trade in the period and receives her continuation value δv_b with probability $1 - \sum_s p_s\beta m_{bs} - \sum_s p_s(1-\beta)m_{sb} = 1 - \sum_s p_s M_{bs}$. The expected future value at the start of a period for a seller equals the maximum of the value of remaining in the market and the value of leaving the market, i.e., $v_s = \max\{v_s(in), 0\}$. The value of remaining in the economy for a seller, $v_s(in)$, is defined similarly to a buyer. Rearranging the equations for $v_k(in)$ gives the following for buyers and sellers:

$$v_s(in) = -c + \sum\nolimits_{b\in B} p_b(\beta t_{bs} + (1-\beta)t_{sb} - M_{bs}(r_s + \delta v_s)) + \delta v_s$$
$$v_b(in) = -c + \sum\nolimits_{s\in S} p_s(M_{bs}(\delta v_{b\cup s} - \delta v_b) - \beta t_{bs} - (1-\beta)t_{sb}) + \delta v_b$$

2.6. **The Competitive Benchmark.** The competitive equilibrium benchmark considered here is a "flow" equilibrium as in Gale (1987) or Satterthwaite and Shneyerov (2007), generalized to accommodate heterogenous goods and multi-unit demand. In each period, flow supply is the measure of sellers of a particular good entering the market and flow demand is the measure of agents willing to purchase a particular good entering the market. In a flow equilibrium, the buyer and seller continuation values, which are the implicit prices, equate flow supply to flow demand for each good that is traded in the market. It is well known that the competitive equilibrium allocations for economy $I \cup S$ are fully characterized by the following linear program (and its dual) which is just the classical Assignment Problem where fractional assignments are permitted. (See, for example, Roth and Sotomayor (1990).) This formulation is a generalization of Shapley and Shubik (1972) to a setting where buyers can purchase multiple commodities as in Kelso and Crawford (1982) or Gul and Stacchetti (1999, 2000).

Primal | **Dual**

$$P = \max_{q \geq 0} \sum\nolimits_{i \in I} \sum\nolimits_{G \subset S} q_{iG}(h_{iG} - \sum\nolimits_{s \in G} r_s) \qquad D = \min_{v \geq 0} \sum\nolimits_{i \in I} v_i + \sum\nolimits_{S} v_s$$

Subject to | Subject to

$$\sum\nolimits_{i \in I} \sum\nolimits_{s \ni G} q_{iG} \leq 1 \text{ for all } s, \qquad v_i + \sum\nolimits_{s \in G} v_s \geq h_{iG} - \sum\nolimits_{s \in G} r_s \ \forall i, G. \tag{5}$$

$$\sum\nolimits_{G \subset S} q_{iG} \leq 1 \text{ for all } i. \tag{6}$$

The vector q that solves the program is a competitive allocation and denotes the measure of matches between buyer i and sellers in the set G that are created in each period of time. Any vector v that solves the dual program is a competitive equilibrium utility vector and the competitive price of a traded good is $p_{x_s} = v_s + r_s$. The constraint given by equation (5) states that the flow demand for sellers of type s, i.e., $\sum_{i \in I} \sum_{s \ni G} q_{iG}$, must be less than the flow supply of that type, which is at most one. This constraint will bind, if the good's price is positive, or more precisely, if $v_s > 0$ and thus $p_{x_s} = v_s + r_s > r_s$. The constraint given by equation (6) states that the flow supply to buyers of type b, must be less than the flow demand by type i, which is at most one. Again, this constraint will bind if $v_i > 0$. Together inequalities (5) and (6) ensure market clearing. Observe that, if q solves the primal and v the dual, then each buyer consumes her most preferred bundle, sellers offer their good only if $p_{x_s} \geq r_s$, and all markets clear. Conversely, if q is a competitive allocation and p a competitive price, then q solves the primal Assignment Problem by the first welfare theorem; and buyer values $v_i = \max_{G \subset S} h_{iG} - \sum_{s \in G} p_{x_s}$ and seller values $v_s = \max\{0, p_{x_s} - r_s\}$ solve the dual Assignment Problem.[7]

3. Convergence to Competitive Equilibria

The development in this section analyzes the limit economy as search becomes costless, i.e., as $\triangle \to 0$. The analysis focuses on sequences of equilibria, the associated sequences of equilibrium match probabilities m^n, type distributions p^n and values v^n and their limit $(\hat{m}, \hat{p}, \hat{v})$. This section's main result, Theorem 1, shows that all trades take place at competitive prices for the economy with agents in the set $I \cup \hat{S}$, asymptotically. The set $\hat{S}$ is defined as the set of markets open at the limit. More precisely, $\hat{S}$ is the set of sellers for whom $\lim_n l_s^n > 0$. In general, $\hat{S}$ need not equal S. That is, $\hat{S}$ may be a proper subset of S, and all markets may not be open at the limit. Consequently, trade may not occur at competitive prices for the economy $I \cup S$. Example 1, at the end of the section outlines such

[7]The substitutes assumption of Kelso and Crawford (1982) or Gul and Stacchetti (1999, 2000) is not needed here. This because there are a continuum of each type and so effectively fractional assignments are allowed.

an example. The corollary to Theorem 1, Corollary 2, assumes that an arbitrarily small fraction of sellers for each good x enter the market in every period. Under this assumption Corollary 1 shows that all trade takes place at competitive prices for the whole economy $I \cup S$, asymptotically.

Let q_{iG} denote the measure of buyers with initial type $i \in I$ leaving the market with bundle G. Also, let

$$e_{iG} = h_{iG} - \sum\nolimits_{s \in G} r_s - \delta \ v_i - \sum\nolimits_{s \in G} \delta v_s$$

denote the *Excess* between any initial buyer type $i \in I$ and sellers in the set G; and similarly

$$e_{bs} = \delta v_{b \cup s} - \delta v_b - \delta v_s - r_s$$

denote the excess between buyer $b \in B$ and seller s.

The proof of Theorem 1 first establishes that the per-period exit rate of buyers with goods in the set G (i.e., $\hat{q}_{iG}$) is a feasible choice for the Assignment Problem, and so, the flow creation of value in the economy is at most as large as the maximized value of the Assignment Problem. The argument proceeds to show that the *Excess* between any initial type i and sellers in the set G (i.e., $\hat{e}_{iG}$) is non-positive. *No Excess* then implies that the vector of equilibrium values v is a feasible choice for the dual of the Assignment Problem, and consequently, that the flow creation of value in the economy is at least as large as the maximized value of the Assignment Problem.[8]

Theorem 1. *Suppose* $(q^n, v^n, p^n) \to (\hat{q}, \hat{v}, \hat{p})$, *let* $\hat{S} = \{s \in S : \limsup l_s^n > 0\}$, *then* $\hat{q}$ *solves the primal Assignment Problem and is a competitive equilibrium allocation for the economy with agents in* $I \cup \hat{S}$; $\hat{v}$ *solves the dual Assignment Problem and is a competitive equilibrium utility vector for the economy with agents in* $I \cup \hat{S}$; *and* $\hat{v}_s + r_s$ *is a competitive equilibrium price for good* x_s.

Proof. Note that $v_k \geq 0$ since k has the option not to enter the market; the best buyer i can do is to consume that agent's favorite bundle G without paying any transfers to any sellers so $v_i \leq \bar{h} = \max_{i,G} h_{iG}$; the best seller j can do is to receive transfer $\bar{h}$ without incurring any costs so $v_j \leq \bar{h}$; and hence $-(|S|+1)(\bar{h}+\bar{r}) \leq e_{iG} \leq \bar{h}$, where $\bar{r} = \max_s r_s$. Below it is shown that $q_{iG} \leq 1$ for all i and G. Consequently, the sequence (q^n, e^n, v^n, p^n) is included in a compact set and has a convergent subsequence. From hereon restrict attention to convergent subsequences $(q^n, e^n, v^n, p^n) \to (\hat{q}, \hat{e}, \hat{v}, \hat{p})$.

Let $b = i_G$ and observe that, q_{iG}, the measure of buyers with initial type $i \in I$ leaving the market with bundle G, is given by the following equation:

$$q_{iG}^n = l_b^n \sigma_b^n(out) + (1 - \sigma_b^n(in)) \sum\nolimits_{s \in G(b)} l_{b \setminus s}^n p_s^n M_{b \setminus ss}^n$$

[8] Observe that the constraint of the dual Assignment Problem only requires *No Excess*.

Note that $l_i = \sum_{G\subset S} l^n_{iG}$ is the measure of buyers, whose initial type was i, present in the market. l_i is in steady state since it is the sum of measure l^n_{iG} that are by assumption in steady state measures. The measure of buyers, whose initial type was i, permanently leaving the market in each period is $\sum_{G\subset S} q^n_{iG}$ and the measure entering is at most 1. Consequently, $\sum_G q^n_{iG} \leq \sigma^n_i(in) \leq 1$.

Note that $\sum_{\{b\in B: s\in G(b)\}} l^n_b + l^n_s$ denotes the measure of agents who own the good that initially belonged to a seller of type s and this measure is in steady state since it is a sum of steady state variables. In each period, the measure of agents leaving with a good that initially belonged to a seller of type s is

$$\sum_{i\in I}\sum_{s\ni G} q^n_{iG} + l^n_s \sigma^n_s(out)$$

and the number of type s agents entering the market is $\sigma_s(in)$. Consequently,

$$\sum_{i\in I}\sum_{s\ni G} q^n_{bG} + l^n_s \sigma^n_s(out) \leq 1.$$

Taking limits shows

$$\sum\nolimits_{i\in I}\sum\nolimits_{s\ni G} \hat{q}_{iG} + \hat{l}_s\hat{\sigma}_s(out) \leq 1 \text{ for all } s \text{ and}$$
$$\sum\nolimits_{G} \hat{q}_{iG} \leq 1 \text{ for all } i.$$

This implies that the vector $\hat{q}$ satisfies equation (5) and equation (6) and is feasible for the primal Assignment Problem. Consequently,

$$\sum\nolimits_{i\in I}\sum\nolimits_{G\subset S} \hat{q}_{iG}(h_{iG} - \sum\nolimits_{s\in G} r_s) \leq P.$$

By Lemma 3, given in the Appendix, $\hat{e}_{iG} \leq 0$ for all i and G , this implies that $\hat{v}$ is feasible for the dual and consequently, $\sum_I \hat{v}_i + \sum_S \hat{v}_s \geq D$. But,

$$\sum_I \hat{v}_i + \sum_S \hat{v}_s \leq \sum\nolimits_{i\in I}\sum\nolimits_{G\subset S} \hat{q}_{bG}(h_{iG} - \sum\nolimits_{s\in G} r_s)$$

by Lemma 4, given in the Appendix. Consequently,

$$D \leq \sum_I \hat{v}_i + \sum_S \hat{v}_s = \sum\nolimits_{i\in I}\sum\nolimits_{G\subset S} \hat{q}_{iG}(h_{iG} - \sum\nolimits_{s\in G} r_s) \leq P = D$$

and so $\hat{q}$ is a competitive allocation and $\hat{v}$ is a competitive equilibrium utility vector. □

Assumption (FD), stated below, posits that the choice of not-entering the market and taking an outside option is not available for an arbitrarily small fraction $\varepsilon_x > 0$ of the lowest cost seller type of each good, at the start of their first period in the market. This choice becomes available only after one period in the market. Consequently, $\varepsilon_x \leq \sum_{x_s=x} l_s$ for all goods x. This assumption ensures that some sellers of each good enter the market and there are no coordination problems in entry that could result in a missing market. If the first draw was not for free, then no agent entering the economy is an equilibrium. Also, Example 1 at the end of the section, outlines a more robust demonstration of a coordination failure.

Assumption. Free First Draw (FD). *In each period, there is $\varepsilon_x > 0$ entry by the lowest cost seller of each good x. These sellers do not pay the cost c in the first period.*

The (FD) assumption allows one to show that, if a buyer waits long enough, then she can meet a seller of any good and make this seller a take-it-or-leave-it offer. This drives the *Excess* between any two agents to zero as search frictions vanish.

Corollary 1. *Assume (FD). If $(q^n, v^n) \to (\hat{q}, \hat{v})$, then $\hat{q}$ solves the primal Assignment Problem and is a competitive equilibrium allocation; $\hat{v}$ solves the dual Assignment Problem and is a competitive equilibrium utility vector; and $\hat{v}_s + r_s$ is a competitive equilibrium price for good x_s.*

Proof. To show convergence, $\hat{e}_{iG} \leq 0$ (no excess) is established for all $G \subset S$. If $\hat{e}_{iG} \leq 0$, then the corollary follows from Theorem 1. Define $\hat{S}$ as in Theorem 1. Let j_x denote the lowest cost seller of good x and observe that $\{j_{x_s}\}_{s\in S} \subset \hat{S}$ by assumption, where $\{j_{x_s}\}_{s\in S}$ is the set of lowest cost sellers in the market. Thus for any $G \subset \{j_{x_s}\}_{s\in S}$, and $i \in I$, $\hat{e}_{iG} \leq 0$.

For any two sellers of good x, $v_s - v_{s'} \leq v_s(in) - v_{s'}(in)$ and so,

$$(v_s(in) - v_{s'}(in))(1-\delta) \leq \beta \sum\nolimits_{b\in B} p_b m_{bs}(r_{s'} + \delta v_{s'} - r_s - \delta v_s) \\ + (1-\beta) \sum\nolimits_{b\in B} p_b m_{sb}(r_{s'} + \delta v_{s'} - r_s - \delta v_s)$$

Also, suppose, without loss of generality, that $r_{s'} \geq r_s$.

$$(v_s - v_{s'})(1-\delta) \leq ((r_{s'} - r_s) - \delta(v_s - v_{s'})) \sum\nolimits_{b\in B} M_{sb}$$
$$v_s - v_{s'} \leq (r_{s'} - r_s) \frac{\sum_{b\in B} M_{sb}}{(1-\delta) + \delta \sum_{b\in B} M_{sb}} \leq r_{s'} - r_s$$

Consequently, $\delta v_s + r_s \leq \delta v_{s'} + r_{s'}$. For any set G of sellers, let H denote the set of sellers where each $s \in G$ is replaced by $j_{x(s)}$, i.e., the lowest cost seller who owns the same good as seller s. So, $h_{iG} = h_{iH}$, also, $\delta v_s + r_s \leq \delta v_{s'} + r_{s'}$ for any $s' \in G$ and $s \in H$ with $x_{s'} = x_s$. Consequently,

$$e_{iG} = h_{iG} - \sum\nolimits_{s\in G} (\delta v_s + r_s) - \delta v_i \leq h_{iH} - \sum\nolimits_{s\in H} (\delta v_s + r_s) - \delta v_i = e_{iH}$$

However, $e^n_{iH} \to \hat{e}_{iH} \leq 0$ since $H \subset \{j_{x_s}\}_{s\in S}$. So, $\hat{e}_{iG} = \lim e^n_{bG} \leq \lim e^n_{iH} \leq 0$ proving that $\hat{e}_{iG} \leq 0$. □

As pointed out the condition outlined in Assumption (FD), or a similar condition imposed on the buyer side of the market, is also necessary in the following limited sense: if

Assumption (FD) does not hold, then there exists a sequence of steady state equilibria for an economy that fails to converge to a competitive equilibrium of that economy. The following is such an example.

Example 1. Necessity of Assumption (FD). *Consider an economy with two buyer types and two seller types, where each buyer wants to purchase only one good and the two seller types own two different goods. Let $h_{12} = h_{21} = 0$ and $h_{11} = h_{22} = 1$, that is h is super-modular; buyer 1 likes seller 1's good and buyer 2 likes seller 2's good. Suppose $r_1 = r_2 = 0$. Let $\delta = 1$. For any $c \leq 1/2$, a unit measure of type 1 buyers and a unit measure of type 1 sellers entering, no type 2 buyers or sellers entering and all meetings resulting in a trade at a price of $1/2$ is an equilibrium. Clearly such a sequence does not converge to the competitive equilibrium of the economy. However, if a tiny fraction ε_2 of type 2 sellers where to enter in each period, then for $c \leq \frac{\varepsilon_2}{1+\varepsilon_2}$ the buyers of type 2 would also find it profitable to enter. This results in the markets for both goods operating and leads to convergence to a competitive equilibrium.*

4. Existence and Characterization of Steady State Search Equilibria

The main result in this section, Theorem 2, shows that, for any configuration of search frictions, that is, for any $\delta \in [0, 1]$ and $c > 0$, a steady state search equilibrium exists. Proposition 2 shows that, in this equilibrium, all proposers make take-it-or-leave-it offers. Also, Corollary 2 to Theorem 2, establishes the existence of full-trade equilibria for small search frictions and shows that these equilibria converge to a competitive equilibrium of the whole economy $I \cup S$.

First, the focus is on the analysis of the three-stage Bayesian game. As stated earlier, in the first stage of the game the proposer chooses a mechanism from the set of feasible mechanisms, in the second stage the responder chooses whether to participate in the mechanism, and in the third stage, if the responder chooses to participate, then the two agents simultaneously report their messages to the mechanism. A mechanism, μ, specifies a set of feasible messages for the proposer and the responder; and for each pair of messages chosen by the two agents, it specifies the probability of trade m and the expected transfer t to be paid by the buyer to the seller. Let $h_{bx} = \delta(v_{b\cup s} - v_b)$ denote the "dynamic" value for a buyer trading with any seller that owns good x, that is, any $s \in S_x = \{s : x_s = x\}$.[9] Also, let $\hat{r}_s = r_s + \delta v_s \geq 0$ denote a seller's "dynamic" reservation value. If a buyer meets a seller of good x, then the buyer's and seller's prior beliefs about the other's type (i.e., the other's dynamic value) is obtained using the steady state measure l; and, in the case of the buyer, conditioning on $s \in S_x$. Once the steady state measure and agent continuation values, and

[9]Observe that $v_{b\cup s} = v_{b\cup s'}$ for any s and $s' \in S_x$.

consequently the dynamic values, are fixed, the three-stage Bayesian game played in each period is identical to the extensive form game analyzed by Maskin and Tirole (1990). Consequently, for any vector of dynamic values and any steady state measure, existence of an perfect Bayesian equilibrium follows from Proposition 6 of Maskin and Tirole (1990), under two additional assumptions retained by Maskin and Tirole (1990).[10] First, *assume that the set of feasible mechanisms is the set of all finite mechanisms where the number of messages available for each type of agent is finite.* Second, *assume that the players have access to a public randomization device in the third stage of the game*, that is, the randomization device is available in the continuation game after a mechanism has been chosen by the proposer and accepted by the responder.[11]

Proposition 1. *For any steady state measure l and any vector of continuation values v, there exists a perfect Bayesian equilibrium for the three-stage mechanism proposal game. Moreover, in this equilibrium all proposers offer the same direct mechanism and all responders accept to participate.*

Proof. Follows immediately from Maskin and Tirole (1990) Proposition 6. □

Having established that an equilibrium exists for the stage game the next proposition characterizes equilibria for the stage game. In particular, the proposition argues that each proposer making a take-it-or-leave-it offer to the responder is an equilibrium of the stage game. Moreover, any equilibrium of the stage-game is payoff equivalent to the equilibrium where every proposer makes a take-or-leave-it offer. This property is a consequence of quasilinear preferences and was established for the case where there are two agent types by Maskin and Tirole (1990) and for the case where there are a continuum of agents and the monotone hazard rate condition is satisfied by Yilankaya (1999). The argument here generalizes the proof in Maskin and Tirole (1990) to the case of arbitrarily many discrete types.

For any steady state measure l and continuation values v pick an equilibrium with strategy and belief profile σ^* and π^* for the three-stage game with the related match probabilities

[10] Maskin and Tirole (1990) consider a model where the number of possible types of the proposer is unrestricted but the number of possible types of the responder is restricted to two. Also in their model utilities for the buyers are strictly concave. However, it can be easily verified that these restrictions are immaterial for their proof of existence.

[11] This randomization device is needed to ensure that the equilibrium payoff set is convex in the third stage after any choice of mechanism. As in Maskin and Tirole (1990) the randomization device is not used on the equilibrium path but is used to support off equilibrium path beliefs. In particular, the randomization device facilitates the coordination of play on a particular Nash equilibrium if the subgame in the third stage, defined by the choice of mechanism, has multiple Nash equilibria.

and expected transfers m^* and t^*.[12] Consider the case when a seller $s \in S_x$ is chosen as the proposer and let V^s denote the expected value for a seller if that seller makes a take-it-or-leave-it offer when that seller is chosen as the proposer. Since the take-it-or-leave-it offer is always available to a seller and can be implemented irrespective of the buyer's beliefs about the seller's type, the expected payoff in the three-stage game when proposing must exceed V^s for all $s \in S_x$, i.e., $\sum_{b\in B} p_b(t^*_{sb} - m^*_{sb}\hat{r}_s) \geq V^s$.

The following optimization maximizes the total expected utility for all sellers of good x ignoring the incentive compatibility constraints for the various seller types. Since the seller incentive constraints are ignored, the mechanism identified by the program might not be chosen by the sellers in the game even though the expected total utility of all seller is maximized by this mechanism.

$$V = \max_{m,t\geq 0} \sum\nolimits_{B\times S_x} p_b l_s(t_{sb} - m_{sb}\hat{r}_s)$$

subject to incentive compatibility and individual rationality:

$$\begin{aligned}
&(IC_{bj}) && \sum\nolimits_S l_s(m_{sj}h_b - t_{sj}) \leq \sum\nolimits_S l_s(m_{sb}h_b - t_{sb})\ \forall b, j \in B && (\alpha_{bj})\\
&(IR) && 0 \leq \sum\nolimits_S l_s(m_{s1}h_1 - t_{s1}) && (\psi)\\
& && m_{sb} \leq 1 && (\gamma_{sb})
\end{aligned}$$

where the Lagrange multipliers are given to the right. The equilibrium match probabilities and expected transfers are incentive compatible and individually rational. So, in particular, they satisfy the constraints of the above maximization problem. This implies that $V \geq \sum_{B\times S_x} l_s p_b(t^*_{sb} - m^*_{sb}\hat{r}_s)$. The following proposition shows that the expected value from all sellers making a take-it-or-leave-it-offer exceeds the value of the above maximization problem, i.e., $\sum_{s\in sS_x} l_s V^s \geq V \geq \sum_{B\times S_x} p_b(t^*_{sb} - m^*_{sb}\hat{r}_s)$. Consequently, the expected utility for each seller $\sum_B p_b(t^*_{sb} - m^*_{sb}\hat{r}_s)$ must equal V^s. Also, since each seller's expected equilibrium payoff in any equilibrium equals V^s, each seller making a take-it-or-leave-it-offer is a perfect Bayesian equilibrium for the stage game, where off-equilibrium path beliefs and play, are given by any other equilibrium belief and strategy profile pair σ^*, π^*.[13]

[12]As a consequence of the Inscrutability Principle of Myerson (1983) all proposers offering the same direct mechanism given by m^* and t^*, all responders accepting and then truthful revelation is also an equilibrium for the three-stage game where equilibrium path beliefs never change and are given by l and off equilibrium path actions and beliefs are given by σ^*, π^*.

[13]The take-it-or-leave-it offer is incentive compatible and individually rational given any equilibrium beliefs for the proposer since it gives the proposer their equilibrium payoff and also is trivially incentive compatible and individually rational for the responder.

Proposition 2. *For any steady state measure l and any vector of continuation values v, in any perfect Bayesian equilibrium of the three-stage game, the proposer's payoff is equal to the payoff the proposer would have received had she made an optimal take-it-or-leave-it offer to the responder. Also, each proposer making an optimal take-it-or-leave-it offer is a perfect Bayesian equilibrium of the three-stage game.*

Proof. The problem for a seller $s \in S_x$, when the seller's reservation value $\hat{r}_s$ is known by the potential buyers, is given by the following maximization problem

$$V^s = \max_{m,t\geq 0} \sum\nolimits_B p_b(t_{bs} - m_{bs}\hat{r}_s)$$

$$(IC^s_{bj}) \qquad h_b m_{js} - t_{js} \leq h_b m_{bs} - t_{bs} \qquad (\alpha^s_{bj})$$

$$(IR^s) \qquad 0 \leq h_1 m_{1s} - t_{1s} \qquad (\psi^s)$$

$$m_{bs} \leq 1 \qquad (\gamma^s_b)$$

Riley and Zeckhauser (1983) showed that a appropriately chosen take-it-or-leave-it offer is optimal when a seller's cost is know and consequently must solve the above program. Let m'_s and t'_s solve problem V^s for each s. The choice $m = (m'_s)$ and $t = (t'_s)$ is feasible for problem V: m'_s and t'_s satisfies the (IC^s) and (IR^s) constraints for each s consequently $m = (m'_s)$ and $t = (t'_s)$ satisfy (IC) and (IR) for problem V, since these constraints are l weighted sums of constraints (IC^s) and (IR^s). This implies that $V \geq \sum_s l_s V^s$. The dual of the linear maximization problem V, is as follows:

$$D = \min_{\gamma\geq 0,\alpha\geq 0,\psi\geq 0} \sum\nolimits_{B\times S_x} l_s\gamma_{sb}$$

subject to:

$$(Mb) \qquad \sum\nolimits_{j\in B}(h_b\alpha_{bj} - h_j\alpha_{jb}) \leq \gamma_{sb} + p_b\hat{r}_s \ \forall b \neq 1, s,$$

$$(Tb) \qquad \sum\nolimits_{j\in B}(\alpha_{jb} - \alpha_{bj}) + p_b \leq 0 \ \forall b \neq 1,$$

$$(M1) \qquad \sum\nolimits_{j\in B}(h_1\alpha_{1j} - h_j\alpha_{j1}) + h_1\psi \leq \gamma_{s1} + p_1\hat{r}_s \ \forall s,$$

$$(T1) \qquad \sum\nolimits_{j\in B}(\alpha_{j1} - \alpha_{1j}) + p_1 \leq \psi.$$

Also, the dual of V^s is as follows

$$D^s = \min_{\gamma\geq 0} \sum\nolimits_b \gamma^s_b$$

subject to:

$$(Mb^s) \qquad \sum\nolimits_{j\in B}(h_b\alpha^s_{bj} - h_j\alpha^s_{jb}) \le p_b\hat{r}_s + \gamma^s_b \forall b \neq 1,$$

$$(Tb^s) \qquad \sum\nolimits_{j\in B}(\alpha^s_{jb} - \alpha^s_{bj}) + p_b \le 0 \forall b \neq 1,$$

$$(M1^s) \qquad \sum\nolimits_{j\in B}(\alpha^s_{j1} - \alpha^s_{1j}) + p_1 \le \psi^s,$$

$$(T1^s) \qquad \sum\nolimits_{j\in B}(h_1\alpha^s_{1j} - h_j\alpha^s_{j1}) + h_1\psi^s \le p_1\hat{r}_s + \gamma^s_1.$$

By Lemma 5 in the appendix, one can pick multipliers $\alpha^s_{bj} = \alpha_{bj}$ for all b, j and s; $\psi^s = 1$ for all s ; and γ^s such that α, ψ and γ^s solve the dual problems D^s for each s. That is, there are dual solutions $\alpha^s, \psi^s, \gamma^s$ for the dual problems D^s where the α^s and the ψ^s components are equal across all s. Observe if α, ψ and γ^s satisfy the constraints (Mb^s) and $(M1^s)$ for all b and s , then α, ψ and γ_s satisfy constraints (Mb) and $(M1)$ for problem (D). Likewise if α and ψ satisfy constraints $(Tb^s\ )$ and $(T1^s)$ for all b and s, then α and ψ satisfy constraints (Tb) and $(T1)$ for problem D. Consequently, α, ψ and $\gamma = (\gamma^s)_{s\in S_x}$ is a feasible solution for D. α, ψ and γ feasible for D implies that $V = D \le \sum_{B\times S_x} l_s\gamma^s_b = \sum_{S_x} l^s D_s = \sum_{S_x} l_s V^s$ and thus proving the result. $\square$

The main theorem, proved in this subsection, establishes that an equilibrium exists, for any $\delta \in [0, 1]$ and $c > 0$. In the model presented here, without an assumption along the lines of (FD), a trivial no-trade equilibrium always exists. Consequently, for a meaningful existence result, the theorem below posits (FD) and establishes the existence of an equilibrium with trade, that is an equilibrium where the markets for all the goods are open. The proof of the theorem involves a straight forward application of Kakutani's fixed point theorem on a mapping defined from the set of feasible measures l, strategy profiles σ and values v, into itself.

Theorem 2. *Assume (FD). For any (c, δ) a search equilibrium (l, σ) exists.*

Example 1 demonstrated that without (FD) there may exist sequences of equilibria that fail to converge to competitive equilibria. The following corollary to Theorem 2 drops Assumption (FD) but maintains the two additional assumptions outlined below. Corollary 2 shows that there exists a sequence of equilibria that converges to a competitive equilibrium for the economy.

The first additional assumption (UNQ), requires that the set of goods traded in any competitive is unique, that is, the same goods are traded in any competitive equilibrium. It should be pointed out that this assumption is automatically satisfied in economies with an homogeneous good such as Gale (1987) and Satterthwaite and Shneyerov (2007), which are special cases of the economy under consideration here. Also, the assumption is satisfied generically for the economies that we consider.[14]

Assumption. Uniqueness (UNQ)**.** *The set of goods traded in any competitive equilibrium is the same. That is if good x is not traded in one competitive equilibrium, then it is not traded in any other CE.*

Assumption (DR) requires the goods in the economy are substitutes for each other from the point of view of all buyers. This assumption is always trivially satisfied in a subset of the economies consider here where buyers have unit demand preferences.

Assumption. Decreasing Returns (DR)**.** *If $G \subset H$, then $h_{bH\cup\{s\}} - h_{bH} \leq h_{bG\cup\{s\}} - h_{bG}$ for all b and s.*

The argument for the corollary is as follows: First fix the set of goods traded in any competitive equilibrium. Assume that a small measure of the lowest cost seller of each of these goods enters in each period, i.e., (FD) holds for the traded goods. Given this assumption a sequence of equilibria, that converges to a competitive equilibrium exists by Corollary 1 and Theorem 2. However, if the measure of sellers with (FD) is picked sufficiently small, then for sufficiently small c^n and $1 - \delta^n$, the measure of sellers of the traded goods entering the economy must exceed the measure of sellers entering due to the (FD) assumption. Consequently, the (FD) assumption is non-binding and can be dropped thus proving the existence of the desired sequence of convergent equilibria. The convergent sequence, however, converges to an competitive equilibrium for the economy where the set of traded goods is a subset of the set of all goods. Assumption (DR) is then used to show that this also a Competitive Equilibrium for the set of all goods.

Corollary 2. *Assume (UNQ) and (DR). There exists a sequence $(q^n, v^n) \to (\hat{q}, \hat{v})$, such that $\hat{q}$ solves the primal Assignment Problem and is a competitive equilibrium allocation; $\hat{v}$ solves the dual Assignment Problem and is a competitive equilibrium utility vector; and $\hat{v}_s + r_s$ is a competitive equilibrium price for good x_s.*

Proof. By (UNQ), the set of goods can be partitioned into two sets $H \subset X$ and $X \setminus H$ where H denotes the set of goods that are traded in any competitive equilibrium. Let q_x denote the measure of good x traded by the lowest cost sellers of good x, i.e., by sellers $S_{\underline{\mathrm{X}}} = \{s : x_s = x \text{ and } r_s \leq r'_s \text{ for all } s' \text{ with } x_{s'} = x\}$, traded in a competitive equilibrium. More precisely

$$q_x = \sum\nolimits_{s \in S_{\underline{\mathrm{X}}}} \sum\nolimits_b \sum\nolimits_{s \ni G} q_{bG}$$

[14]This is because the linear program that characterizes the set of competitive allocations generically has a unique solution.

Also, let $\underline{q}_x = min_{q \in Q} q_x$ where Q denotes the set of competitive allocations. Note that Q is a compact and convex set and $\underline{q}_x > 0$ for any $x \in H$. Assume (FD) for all $x \in H$ and let the measure of low cost sellers of good $x \in H$ receiving the first draw free be $0 < \varepsilon_x < \underline{q}_x$. Observe that given this set-up, the sequence of equilibria will converge to $\hat{q}$, which is competitive equilibrium for the economy comprised of sellers such that $x_s \in H$ and $b \in B$. Also, observe that since only goods in H are traded, q is also an efficient allocation for the original economy I. For any buyer b with $\hat{l_b} > 0$, $\hat{e}_{bG} \leq 0$ for any $G \subset S$. For any buyer with $\hat{l_b} = 0$, $\hat{e}_{bG} \leq 0$ for any $G \subset \{s : x_s \in X \setminus H\}$. This is because otherwise, i.e., is $\hat{e}_{bG} > 0$, then allocating to b, who is not trading, the goods in G, which are not being traded, would improve the efficiency of the matching which would contradict that the matching $\hat{q}$ is efficient. So $\hat{e}_{bG} \leq 0$ for $G \subset \{s : x_s \in X \setminus H\}$. Also, for $\hat{l_b} = 0$, $\hat{e}_{bG} \leq 0$ for any $G \subset \{s : x_s \in H\}$. But, $\hat{e}_{bG} \leq 0$ for $G \subset \{s : x_s \in H\}$ and $G \subset \{s : x_s \in X \setminus H\}$ in conjunction with (DR) implies that $\hat{e}_{bG} \leq 0$ for all b and $G \subset S$. This, in turn, shows that the allocation $\hat{q}$ is a competitive equilibrium allocation for I and $\hat{v}$ is a competitive utility vector.

Now observe that for sufficiently large n, $\sigma^n_{s_x}(in) > \varepsilon_x$ since the measure of lowest cost sellers leaving the market must converge to competitive competitive equilibrium which exceeds $\underline{q}_x$. This implies that for n sufficiently large $v_s(in) \geq 0$. This shows that we can drop the (FD) assumption which is not binding for sufficiently large n and just take entry by type s_x to equal $\sigma^n_{s_x}(in)$. □

5. Discussion and Conclusion

This paper presented a model where buyers purchase a bundle of indivisible, heterogeneous goods from sellers who are each endowed with one unit of a good. Trade takes place in a decentralized market under two sided incomplete information. A small measure of the lowest cost seller of each good is assumed to sample the market at least once. Under this assumption an equilibrium is shown to exist (Theorem 2) and any sequence of equilibria is shown to converge to a competitive equilibrium.

The model presented here considered the case where agents bargain pairwise, where as other studies in the literature, such as Satterthwaite and Shneyerov (2007), analyze bargaining in larger coalitions. The convergence result is not sensitive to this assumption. In particular, the results presented here are robust to any random matching technology as long as any buyer and seller whose exist with positive measure in the economy meet with positive probability. Also, the analysis proceeded under the assumption of two sided incomplete information. However, all the results presented also go through without alteration under complete information. Finally, a central assumption in the model maintained throughout the paper was that the economy remains in steady state. An immediate way to extend this model is to drop the steady state assumption and consider a non-stationary market with finitely many, instead of a continuum, of agents entering in each period. Under such a formulation, the goal would be to show that trade always occurs at competitive prices and that the market clears on average.

Appendix A. Omitted Proofs

A.1. **Proof of Theorem 1.**

Lemma 1 (No Excess 1)**.** *If* $\max\{\hat{p}_b, \hat{p}_s\} > 0$*, then* $e_{bs} \leq 0$*.*

Proof. For any c and δ a seller (or buyer) can offer to sell her good for $\delta v_{b\cup s} - \delta v_b - \varepsilon$ and ensure that buyer b purchases if they meet, since the payoff that buyer b gets from purchasing the good strictly exceeds her continuation payoff $\delta^n v_b$. Also, any buyer can offer to buy a good for $r_s + \delta v_s + \varepsilon$, and ensure that she makes a purchase if she meets seller s. Consequently, for any c and δ

$$\begin{aligned} v_s &\geq -c + (1-\beta)p_b(\delta v_{b\cup s} - \delta v_b - r_s) + \delta(1-(1-\beta)p_b)v_s \\ (1-\delta)v_s &\geq -c + (1-\beta)p_b(\delta v_{b\cup s} - \delta v_b - \delta v_s - r_s), \text{ and} \\ (1-\delta)v_b &\geq -c + \beta p_s(\delta v_{b\cup s} - \delta v_b - \delta v_s - r_s). \end{aligned}$$

So

$$\begin{aligned} \lim_n (1-\delta^n)v_s^n &\geq \lim_n -c^n + (1-\beta)p_b^n e_{bs}^n, \\ \lim_n (1-\delta^n)v_b^n &\geq \lim_n -c^n + \beta p_s^n e_{bs}^n. \end{aligned}$$

Taking limits shows that $\hat{p}_b \hat{e}_{bs} \leq 0$ and $\hat{p}_s \hat{e}_{bs} \leq 0$. However, since $\max\{\hat{p}_b, \hat{p}_s\} > 0$, $\hat{e}_{bs} \leq 0$. □

Lemma 2. *Let* $L^n = \max\{L_B^n, L_S^n\}$, $\lim_n c^n L^n = 0$ *and* $\lim_n (1-\delta^n)L^n = 0$.

Proof. If $\limsup L^n < \infty$, then since $0 \leq L^n$, $\lim_n c^n L^n = 0$ and $\lim_n (1-\delta^n)L^n = 0$. If $\limsup L^n = \infty$, then either (inclusive) $\limsup L_b^n = \infty$ or $\limsup L_s^n = \infty$. Assume, $\limsup L^n = \infty$ and, without loss of generality, that $\limsup L_b^n \geq \limsup L_s^n$. For $l_b > 0$, $v_b = v_b(in)$ and thus the following equation is satisfied for all $b \in B$,

$$(1-\delta)l_b v_b + l_b c = \sum\nolimits_{s\in S} l_b p_s (M_{bs}(\delta v_{b\cup s} - \delta v_b) - \beta t_{bs} - (1-\beta)t_{sb})$$

Observing that $0 \leq (1-\delta)l_b v_b$ and summing up over all buyers in B gives

$$\sum_{b\in B} l_b c \leq L \sum_{b\in B} p_b \sum_{s\in S} p_s (M_{bs}(\delta v_{b\cup s} - \delta v_b) - \beta t_{bs} - (1-\beta)t_{sb})$$

Individual rationality for any seller s implies that $\sum_B p_b(\beta t_{bs}+(1-\beta)t_{sb}-M_{bs}(r_s+\delta v_s)) \geq 0$ since otherwise this seller would do strictly better off by not trading. Substituting this for transfers above gives

$$\begin{aligned}\sum_{b\in B} l_b c &\leq L\sum_{B\times S} p_b p_s M_{bs}(\delta v_{b\cup s}-\delta v_b-(r_s+\delta v_s))\\ &= \sum_{\hat{B}\times S} l_b p_s M_{bs} e_{bs} + \sum_{(B\setminus\hat{B})\times S} l_b p_s M_{bs} e_{bs}\end{aligned}$$

where $\hat{B}=\{B:\hat{p}_b>0\}$. In Lemma 1 it was shown that $(1-\delta)v_b+c\geq \beta p_s e_{bs}$ for each b and also $v_b\leq \bar{h}$ consequently

$$\sum_{b\in B} l_b c \leq \sum_{\hat{B}\times S} l_b p_s M_{bs} e_{bs} + \sum_{B\setminus\hat{B}} \frac{l_b|S|}{\beta}(c+(1-\delta)\bar{h}).$$

Observe that $\lim \frac{1-\delta^n}{c^n} = \lim_{\Delta t\to 0}\frac{1-e^{-\rho\Delta t}}{\Delta t\kappa} = \frac{\rho}{\kappa} > 0$. Consequently, for Δt sufficiently small there exists ξ such that $c+(1-\delta)\bar{h}\leq c(1+\xi)$. Substituting yields

$$\sum_{b\in B} l_b c \leq \sum_{\hat{B}\times S} l_b p_s M_{bs} e_{bs} + \sum_{B\setminus\hat{B}} l_b c\frac{|S|(1+\xi)}{\beta}$$

$$\left(\sum_{b\in B} l_b c\right)\left(1-\frac{Lc\sum_{B\setminus\hat{B}} p_b\frac{|S|(1+\xi)}{\beta}}{Lc\sum_{b\in B} p_b}\right) \leq \sum_{\hat{B}\times S} l_b p_s M_{bs} e_{bs}$$

$$\left(\sum_{b\in B} l_b c\right)\left(1-\frac{\sum_{B\setminus\hat{B}} p_b\frac{|S|(1+\xi)}{\beta}}{\sum_{b\in B} p_b}\right) \leq \sum_{\hat{B}\times S} l_b p_s M_{bs} e_{bs}$$

The assumption of steady state implies that $\sum_s l_b p_s M_{bs}\leq 1$ for each b and consequently,

$$\left(\sum_{b\in B} l_b c\right)\left(1-\frac{\sum_{B\setminus\hat{B}} p_b\frac{|S|(1+\xi)}{\beta}}{\sum_{b\in B} p_b}\right) \leq \sum_{\hat{B}\times S} e_{bs}$$

By the definition of $\hat{B}$, $\lim_n \sum_{B\setminus\hat{B}} p_b^n = 0$, $\lim_n \sum_{b\in B} p_b^n = 1$ and $\lim_n 1-\frac{\sum_{B\setminus\hat{B}} p_b^n\frac{|S|(1+\xi)}{\beta}}{\sum_{b\in B} p_b^n} = 1$ and consequently,

$$\lim_n\left(\sum_{b\in B} l_b^n c^n\right)\lim_n\left(1-\frac{\sum_{B\setminus\hat{B}} p_b^n\frac{|S|(1+\xi)}{\beta}}{\sum_{b\in B} p_b^n}\right) \leq \lim_n \sum_{\hat{B}\times S} e_{bs}^n$$

$$\lim_n \sum_{b\in B} l_b^n c^n \leq \lim_n \sum_{\hat{B}\times S} e_{bs}^n$$

However, if $\hat{p}_b > 0$, then $\lim_n e^n_{bs} = 0$ for any s , by Lemma 1. This implies that

$$0 \leq \lim_n \sum_{b \in B} l^n_b c^n \leq \lim_n \sum_{\hat{B} \times S} e^n_{bs} = 0$$

Also, $\lim L^n c^n = 0$, implies, $\lim_n (1-\delta^n) L^n = 0$ because $\lim \frac{1-\delta^n}{c^n} = \frac{\rho}{\kappa}$. □

Lemma 3. *$\hat{e}_{iG} \leq 0$ for all $i \in I$ and $G \subset \hat{S}$.*

Proof. By the argument provided in Lemma 1, for $i \in I$,

$$(1-\delta^n) v^n_i \geq -c^n + \beta p^n_{s1} (\delta^n v_{i_{\{s\}}} - \delta^n v^n_i - \delta^n v^n_s - r^n_s).$$

Multiply both sides by $L^n = \max\{L^n_B, L^n_S\}$ which gives

$$((1-\delta^n) v^n_i + c^n) L^n \geq \beta L^n p^n_{s1} (\delta^n v^n_{i_{\{s1\}}} - \delta^n v^n_{ii} - \delta^n v^n_{s1} - r^n_{s1}).$$

However, by Lemma 2,

$$\lim_n ((1-\delta^n) v^n_i + c^n) L^n = 0.$$

Also, by assumption, $\lim L^n p^n_s = \lim l^n_s > 0$ for all $s \in \hat{S}$, and consequently,

$$\hat{v}_{i_{\{s\}}} - \hat{v}_b - \hat{v}_s - r_s \leq 0.$$

Also, again by the argument provided in Lemma 1,

$$((1-\delta^n) v^n_{i_{\{s,s2\}}} + c^n) L^n \geq \beta L^n p^n_{s2} (\delta^n v_{i_{\{s,s2\}}} - \delta^n v^n_{i\{s\}} - \delta^n v^n_{s2} - r^n_{s2}).$$

So, $\hat{v}_{i_{\{s,s2\}}} - \hat{v}_{i_{\{s\}}} - \hat{v}_{s2} - r_{s2} \leq 0$. Substituting gives

$$\hat{v}_{i_{\{s,s2\}}} - \hat{v}_i - \hat{v}_{s2} - \hat{v}_s - r_s - r_{s2} \leq 0$$

Repeating $|G|$ times shows that

$$\hat{v}_{i_G} - \hat{v}_i - \sum\nolimits_{s \in G} (\hat{v}_s + r_s) \leq 0.$$

However, $v^n_{iG} \geq h_{iG}$ for all n and so $\hat{v}_{i_G} \geq h_{iG}$. Thus

$$h_{iG} - \hat{v}_i - \sum\nolimits_{s \in G} (\hat{v}_s + r_s) \leq 0$$

proving the result. □

Lemma 4. $\sum_{i \in I} \hat{v}_i + \sum_S \hat{v}_s \leq \sum_{i \in I} \sum_G \hat{q}_{iG} (h_{iG} - \sum_{s \in G} r_s)$.

Proof. The value equations for the buyers implies

$$l_b v_b (1-\delta) \leq \beta \sum\nolimits_s l_b p_s m_{bs} (\sigma_{b \cup s}(in) \delta (v_{b \cup s} - h_{i(b)G(b \cup s)}) + \delta h_{i(b)G(b \cup s)} - \delta v_b - t_{bs})$$
$$+ (1-\beta) \sum\nolimits_s l_b p_s m_{sb} (\sigma_{b \cup s}(in) \delta (v_{b \cup s} - h_{i(b)G(b \cup s)}) + \delta h_{i(b)G(b \cup s)} - \delta v_b - t_{sb})$$

Summing up over all buyers and taking the limit as $\delta \to 1$ and observing that t_{bs} goes to $v_s + r_s$ for any b and s with $m_{bs} > 0$ gives

$$0 \leq \sum\nolimits_{b\in B} \sum\nolimits_{s\in S}(1-\hat{\sigma}_{b\cup s}\,(in))(\beta\hat{l}_b\hat{p}_s\hat{m}_{bs}+(1-\beta)\hat{l}_b\hat{p}_s\hat{m}_{sb})(h_{i(b)G(b\cup s)}-\hat{v}_{i(b)}-\sum\nolimits_{j\in G(b\cup s)}(\hat{v}_j+r_j))$$

rearranging shows that

$$0 \leq \sum\nolimits_{b\in B} \sum\nolimits_{s\in S}(1-\hat{\sigma}_b(in))(\beta\hat{l}_{b\setminus s}\hat{p}_s\hat{m}_{b\setminus ss}+(1-\beta)\hat{l}_{b\setminus s}\hat{p}_s\hat{m}_{sb\setminus s})(h_{i(b)G(b)}-\hat{v}_{i(b)}-\sum\nolimits_{j\in G}(\hat{v}_j+r_j))$$

However

$$\sum\nolimits_{s\in S}(1-\hat{\sigma}_{b_G}(in))(\beta\hat{l}_{b_{G\setminus s}}\hat{p}_s\hat{m}_{b_{G\setminus s}s} + (1-\beta)\hat{l}_{b_{G\setminus s}}\hat{p}_s\hat{m}_{sb_{G\setminus s}}) = \hat{q}_{i(b)G(b)}$$

which implies that

$$0 \leq \sum\nolimits_{b\in B} \hat{q}_{i(b)G(b)}(h_{i(b)G(b)} - \hat{v}_{i(b)} - \sum\nolimits_{j\in G(b)}(\hat{v}_j + r_j))$$
$$0 \leq \sum\nolimits_{i\in I} \sum\nolimits_{G} \hat{q}_{iG}(h_{iG} - \hat{v}_i - \sum\nolimits_{j\in G}(\hat{v}_j + r_j))$$

Observe that for i with $\hat{v}_i > 0$ $\sum_G \hat{q}_{iG} = 1$ and for s with $\hat{v}_s > 0$, $\sum_b \sum_{s\ni G} \hat{q}_{bG} = 1$ so

$$\sum\nolimits_{i\in I} \hat{v}_i + \sum\nolimits_{S} \hat{v}_s \leq \sum\nolimits_{i\in I} \sum\nolimits_{G} \hat{q}_{iG}(h_{iG} - \sum\nolimits_{s\in G} r_s)$$

proving the result. □

A.2. **Proof of Proposition 2.** Let the buyers in the set B are arranged in decreasing order of their dynamic value with h_1 being the lowest and $h_{|B|}$ the highest dynamic buyer value. For the proposition we need to show that there exists α, ψ and $(\gamma^s)_{s\in S_x}$ such that α, ψ and γ^s solve D_s for each s where the α and ψ are independent of s . In the case where the marginal revenue is decreasing, this is straight forward. Suppose that the marginal revenue is increasing in b for all $b \in B$, that is,

$$\Delta_b = h_b - (h_{b+1} - h_b)(\sum_{k=b+1}^{|B|} p_k)/p_b$$

is increasing in b. Set $\psi^s = \psi = \sum_{b=1}^{|B|} p_b = 1$ for all s; $\alpha^s_{b,b-1} = \alpha_{b,b-1} = \sum_{k=b}^{|B|} p_k$ and all other $\alpha_{bk} = 0$ for all s; and $\gamma^s_b = p_b \max\{\Delta_b - \hat{r}_s, 0\}$. Observe that

$$\sum_{j\in B}(h_b\alpha^s_{bj} - h_j\alpha^s_{jb}) = h_b\alpha_{b,b-1} - h_{b+1}\alpha_{b+1,b} = h_b\sum_{k=b}^{|B|} p_k - h_{b+1}\sum_{k=b+1}^{|B|} p_k = p_b\Delta_b$$

Also,

$$p_b\hat{r}_s + \gamma^s_b = p_b\max\{\Delta_b - \hat{r}_s, 0\} + p_b\hat{r}_s = p_b\max\{\Delta_b, \hat{r}_s\}$$

Consequently,

$$\sum_{j\in B}(h_b\alpha_{bj}^s - h_j\alpha_{jb}^s) = p_b\Delta_b \le p_b\hat{r}_s + \gamma_b^s = p_b \max\{\Delta_b, \hat{r}_s\}$$

and so (Mb^s) is satisfied.

Also,

$$\sum\nolimits_{j\in B}(\alpha_{jb}^s - \alpha_{bj}^s) + p_b = \alpha_{b+1,b} - \alpha_{b,b-1} + p_b = \sum_{k=b+1}^{|B|} p_k - \sum_{k=b}^{|B|} p_k + p_b = 0$$

and so (Tb^s) is satisfied. Consequently, these multipliers satisfy all constraints for the dual optimization problems D^s.

In the case where the marginal revenue is not necessarily decreasing, the argument is more involved. The following lemma proves the general result.

Lemma 5. *There exists α, ψ and $(\gamma^s)_{s\in S_x}$ such that α, ψ and γ^s solve D^s for each s.*

Proof. Observe that the optimal solution to each V^s is a take it or leave it offer. In an optimum for the primal optimization problem V^s the upward incentive constraints, i.e, (IC_{jb}^s) where $j < b$, never bind so always take $\alpha_{jb} = 0$ if $j < b$.

The proof is by induction. Take $\psi = 1$. Pick α such that $(\alpha, 1, \gamma^s) \in \arg\min D^s$ for all $s = 2, ..., |S_x|$ with $r_2 < r_3... \le r_{|S_x|}$. I show that there exists α' such that $(\alpha', 1, \gamma^s) \in \arg\min D^s$ for all $s = 1, 2, ..., |S_x|$ with $r_1 < r_2 < r_3... \le r_{|S_x|}$.

Let $b(s) \equiv \min_b\{m_b^s > 0, (m,t) \in \arg\max V^s\}$. Choose $(\alpha, 1, \gamma^s) \in \arg\min D^s$ $s = 2, ..., |S_x|$ such that the primal solution has $m_{b(2)}^2 > 0$. Observe for $b > b(2)$ and $k < b(2)$, $\alpha_{bk} = 0$. This is because α_{bk} is the multiplier associated with IC_{bk}^s which holds strictly. Also, let $\alpha_{b(2)k} = 0$ for all $k > 1$ and let $\alpha_{bk} = 0$ for all $b < b(2)$ and $k \ge 1$. This works because all constraints $(IC_{b(2)k}^2)$ are identical and have righthand sides equal to zero, and all constraints (IC_{bk}^2) for all $b < b(2)$ and $k \ge 1$ the same and have both left and righthand sides equal to zero. Summing over all constraints (Tb^2) (which hold strictly since they are the dual constraint associated with transfer payments T) gives

$$\sum_{b>b(2)} (p_b + \sum_{k>b} \alpha_{kb} - \sum_{k<b} \alpha_{bk}) + p_{b(2)} + \sum_{k>b(2)} \alpha_{kb(2)} - \alpha_{b(2),1} = 0$$

$$\alpha_{b(2),1} = \sum_{b\ge b(2)} p_b.$$

For $b < b(2)$, $\gamma_b^s = 0$ since this is the multiplier on constraint $Mb^s \le 1$. Also, for $b \ge b(2)$

$$\sum_{k<b} h_b\alpha_{bk} - \sum_{k>b} h_k\alpha_{kb} = p_b r_s + \gamma_b^s$$

since the multiplier $Mb^s > 0$ for all $b \geq b(2)$.

Claim 1. There exists $(\alpha^1, 1, \gamma') \in \arg\min D^1$ such that $\alpha^1_{bj} = \alpha_{bj}$ for $b \geq b(2)$ and $j > b(2)$ and $\gamma_b = \gamma^2_b + (r_2 - r_1)p_b$ for $b \geq b(2)$.

Proof. Take the alternative maximization problem where all buyers with valuation $h_b \geq h_{b(2)}$ are viewed as one buyer with mass $\sum_{b \geq b(2)} p(b)$ and valuation $h_{b(2)}$. This maximization has an identical solution (to the maximization problem under consideration) for seller r_1. Pick the multipliers α^1_{bj} for $b \leq b(2)$ and $j < b(2)$ and γ'_b for $b < b(2)$ using the alternative maximization problem and set $\alpha^1_{bj} = \alpha_{bj}$ for $b \geq b(2)$ and $j > b(2)$ and $\gamma'_b = \gamma^2_b + (r_2 - r_1)p_b$ for $b \geq b(2)$. These multipliers satisfy dual constraints and solve the dual maximization problem. □

Claim 2. Pick $(\alpha^1, 1, \gamma') \in \arg\min D^1$ from Claim 1. There exists $0 \leq \gamma^1_b$ such that $\gamma^1_b \leq p_b \min(h_b - r_1, r_2 - r_1)$ for all $b(1) \leq b < b(2)$,

$$\sum_{k=b}^{b(2)-1} \gamma^1_k \geq (h_b - r_1) \sum_{k=b}^{b(2)-1} p_k - (h_{b(2)} - h_b)(\sum_{k \geq b(2)} p_k)$$

for all $b(1) \leq b < b(2)$. Also, for $(\alpha^1, 1, \gamma') \in \arg\min D^1$ from Claim 1, $\gamma^1_k = \gamma'_k$ for $k \geq b(2)$ and $\sum_k \gamma'_k = \sum_k \gamma^1_k$.

Proof. Pick $(\alpha^1, 1, \gamma') \in \arg\min D^1$ from Claim 1. Observe that $(h_b - r_2)\sum_{k=b}^{b(2)-1} p_k - (h_{b(2)} - h_b)(\sum_{k=b(2)}^{|B|} p_k) < 0$ for any $b < b(2)$ because otherwise $b(2)$ would not be the cut-off type for seller with cost r_2. Shuffle γ^1_b for $b = b(2)-1, ..., b(1)$ to ensure the above inequalities hold. □

Let

$$\Delta_b(\alpha) = \sum_{k<b} h_b \alpha_{bk} - \sum_{k>b} \alpha_{kb} h_k - p_b r_1.$$

Observing that $\sum_{k<b} \alpha_{bk} = p_b + \sum_{k>b} \alpha_{kb}$, gives

$$\Delta_b(\alpha) = p_b(h_b - r_1) - \sum_{k=b+1}^{b(2)} (h_k - h_b)\alpha_{kb}.$$

Claim 3. There exists $(\alpha', 1, \gamma^s) \in \arg\min D^s$ for $s = 2, ..., |S_x|$ such that $\gamma^1_{b(2)-1}$ identified in Claim 2 with α' satisfy dual constraints $(Mb(2) - 1^1)$ and $(Tb(2) - 1^1)$ for problem D^1. Also, this process can be repeated so that γ^1 and α' satisfy dual constraints (Mk^1) and (Tk^1) for all k for problem D^1.

Proof. Pick (as above) $(\alpha, 1, \gamma^s) \in \arg\min D^s \ s = 2, ..., |S_x|$. Let $b = b(2) - 1$. Observe that $\alpha_{b(2),1} = \sum_{k \geq b(2)} p_k$. For $z_b \in [0,1]$ set $\alpha'_{b(2)b} = z_b \alpha_{b(2)1}$. So

$$\begin{aligned} \Delta_b(\alpha') &= \sum_{k<b} h_b \alpha'_{bk} - \sum_{k>b} \alpha'_{kb} h_k - p_b r_1 \\ \Delta_b(\alpha'(z_b)) &= p_b(h_b - r_1) - z_b(\sum_{k \geq b(2)} p_k)(h_{b(2)} - h_b). \end{aligned}$$

The definition of $\Delta_b(\alpha'(z_b))$ presumes that (Tb^1) hold with equality. Observe that if $z_b = 0$, then $\Delta_b(\alpha'(0)) = p_b(h_b - r_1) \geq p_b \min\{r_2 - r_1, h_b - r_1\} \geq \gamma_b^1$ and $\frac{\partial \Delta_b(\alpha'(0))}{\partial z_b} < 0$. Also,

$$\Delta_b(\alpha'(1)) = (h_b - r_1)p_b - (\sum_{k \geq b(2)} p_k)(h_{b(2)} - h_b) \leq \gamma_b^1$$

by Claim 2. Consequently, there exists $z_b \in [0,1]$ such that $\Delta_b(\alpha'(z_b)) = \gamma_b^1$. For $b-1$ set $\alpha'_{b(2),b-1} = z_{b-1}(1 - z_b)\alpha_{b(2)1}$, $\alpha'_{b,b-1} = z_{b-1}(p_b + z_b \alpha_{b(2),1})$ and let $\sum_{k<b-1} \alpha'_{b-1,k} = p_{b-1} + z_{b-1}(p_b + \alpha_{b(2)1})$ so that constraint (Tb^1) holds. Again, Claim 2 can be used to pick $z_k \in [0,1]$.

Define $\alpha'_{kb(1)} = p_k + \sum_{j>k} \alpha'_{jk} - \sum_{b(1)<j<k} \alpha'_{kj}$ and $\alpha'_{b(1),1} = p_{b(1)} + \sum_{k>b(1)} \alpha'_{kb(1)}$ ensuring that (Tb^1) hold with equality. Observe that $\alpha'_{b(1),1} = p_{b(1)} + \sum_{k>b(1)} \alpha'_{kb(1)} = \sum_{b \geq b(1)} p_b$. By Claim 2

$$\begin{aligned} \sum_{k=b(1)}^{b(2)-1} \gamma_k^1 &= (\sum_{k=b(1)}^{b(2)-1} p_k) h_{b(1)} - r_1(\sum_{k=b(1)}^{b(2)-1} p_k) - (\sum_{k \geq b(2)} p_k)(h_{b(2)} - h_{b(1)}) \\ &= (\sum_{k \geq b(1)} p_k) h_{b(1)} - r_1(\sum_{k=b(1)}^{b(2)-1} p_k) - (\sum_{k \geq b(2)} p_k) h_{b(2)} \end{aligned}$$

Also, by the choice of α^1,

$$\sum_{k=b(1)+1}^{b(2)-1} \gamma_k^1 = \sum_{k=b(1)+1}^{b(2)-1} \alpha'_{kb(1)} h_k - (\sum_{k=b(1)+1}^{b(2)-1} \alpha'_{b(2)k}) h_{b(2)} - r_1 \sum_{k=b(1)+1}^{b(2)-1} p_k$$

Consequently,

$$\begin{aligned}\gamma^1_{b(1)} &= \sum_{k=b(1)}^{b(2)-1}\gamma^1_k - \sum_{k=b(1)+1}^{b(2)-1}\gamma^1_k \\ &= (\sum_{k\geq b(1)} p_k)h_{b(1)} - \sum_{k=b(1)+1}^{b(2)-1}\alpha'_{kb(1)}h_k - h_{b(2)}(\sum_{k\geq b(2)} p_k - (\sum_{k=b(1)+1}^{b(2)-1}\alpha'_{b(2)k})) - r_1 p_{b(1)} \\ &= (\sum_{k\geq b(1)} p_k)h_{b(1)} - \sum_{k=b(1)+1}^{b(2)-1}\alpha'_{kb(1)}h_k - h_{b(2)}\alpha'_{b(2)b(1)} - r_1 p_{b(1)} = \Delta_{b(1)}(\alpha')\end{aligned}$$

proving the result. □

The $\alpha', 1, \gamma^s$ identified by the previous claim is an element of $\arg\max D^s$ for $s = 1, ..., |S_x|$ since $\sum \gamma^1_b = (h_{b(1)} - r_1)\sum p_b$ and α' and γ^1 satisfy all constraints of the problem D^1 thus completing the proof. □

A.3. **Proof of Theorem 2.**

Proof. The argument identifies a candidate equilibrium where each proposer chooses an optimal mechanism as if their type is observed (the take-it-or-leave-it offer is a optimal choice), and the steady state measure and values are consistent with the matching probabilities and transfer payments generated by these optimal mechanisms. The candidate equilibrium is generated via a fixed point argument outlined below. Proposition 2 implies that the take-it-or-leave offer that solves the mechanism choice problem where the proposer's type is observed (or any other solution) is also a solution to the mechanism choice problem where the proposer's type is private information for appropriately chosen off-equilibrium path beliefs π. Consequently, the candidate equilibrium is an equilibrium for the economy with the off equilibrium path beliefs given by π.

For any c and δ, $0 \leq v_i \leq \bar{h}$. Let $V = \{v \in R^{|I|\times 2^{|S|}+|S|} : 0 \leq v_i \leq \bar{h}\}$ denote the set of possible values.

Let $\epsilon_s = \epsilon_x$, if s is the lowest cost seller of good x_s and $\epsilon_s = 0$ otherwise. For all $i \in I$ and $s \in S$, if $v_s(in) < 0$, then $l_s = \epsilon_s$ by Assumption (FD), and if $v_i(in) < 0$, then $l_i = 0$. If $v_s(in) \geq 0$, then $l_s = 1/(\sum_{b\in T} p_b M_{bs} + \sigma_s(out))$. For $i \in I$, if $v_i(in) \geq 0$, then $l_i = 1/(\sum_{s\in S} M_{is} + \sigma_i(out))$. Also, if $\sigma_b(in)\sum_{s\in G(b)} l_{b\setminus s}M_{b\setminus ss} \geq 0$, then

$$l_b = \frac{\sigma_b(in)\sum_{s\in G(b)} l_{b\setminus s}p_s M_{b\setminus ss}}{\sum_{s\in S} p_s M_{bs} + \sigma_b(out)}.$$

Observe that $v_j(in) \leq -c + \sum_k p_k M_{jk}\bar{h}$ and so $c/\bar{h} \leq \sum_k p_k M_{jk}$. Consequently, $1 \leq l_j \leq \frac{\bar{h}}{c}$. Let $\Lambda = \{l : 1 \leq l_j \leq \frac{\bar{h}}{c}\}$ denote the set of possible steady state measures. Let m_{bs} and transfer t_{bs} be the mechanism choice by the buyers and m_{sb} and t_{sb} the mechanism choice by the sellers, and $\sigma_i = (\sigma_{i0}, \mu_i)$. Start with any $l \in \Lambda$, $\sigma \in \Sigma$, $v \in V$ and let

$$\begin{aligned} l'_s(l,\sigma,v) =& \frac{\max\{\sigma_s(in), \epsilon_s\}}{\max\{c/\bar{h}, \sum_{b\in B} p_b M_{bs}\}}, \\ l'_i(l,\sigma,v) =& \frac{\sigma_b(in)}{\max\{c/\bar{h}, \sum_{s\in S} p_s M_{is}\}} \text{ for } i \in I \text{ and,} \\ l'_b(l,\sigma,v) =& \frac{\sigma_b(in)\sum_{s\in G(b)} l_{b\backslash s} p_s M_{b\backslash ss}}{\max\{c/\bar{h}, \sum_{s\in S} p_s M_{bs}\}} \text{ for } b \in B \end{aligned}$$

where the M's are calculated according to σ. This defines a continuous function from $\Lambda \times \Sigma \times V$ into Λ, where $(l,\sigma,v) \mapsto l'_i$ for each i.

Let

$$\begin{aligned} v'_b(in|l,\sigma,v) = \max\nolimits_{(m'_b,t'_b)\geq 0} -c + \beta \sum p_s(m'_{bs}(\delta v_{b\cup s} - \delta v_b) - t'_{bs}) \\ + (1-\beta)\sum_s p_s(m_{sb}(\delta v_{b\cup s} - \delta v_b) - t_{sb}) + \delta v_b \end{aligned}$$

subject to

$$\begin{aligned} t'_{bs} - m'_{bs}(r_s + \delta v_s) &\geq t'_{bj} - m'_{bj}(r_s - \delta v_s) \text{ for all } s \text{ and } j \in S \\ t'_{bs} - m'_{bs}(r_s + \delta v_s) &\geq 0 \text{ for all } s \\ m'_{bs} &\leq 1 \text{ for all } s. \end{aligned}$$

Also, let $S'_{b,1}(l,\sigma,v)$ denote the set of maximizers for the above program. Observe that the objective function of this maximization problem is continuous and concave in m and t, and the constraint set, defined by linear inequalities, is non-empty ($m = 0$ and $t = 0$ is always feasible), convex and compact for any choice of $(l,\sigma,v) \in \Lambda \times \Sigma \times V$. Consequently, $v'_b(in|l,\sigma,v)$ is a continuous function of (l,σ,v) and $S'_{b,1}(l,\sigma,v)$ is a upper-hemi-continuous (UHC), non-empty, convex and compact valued correspondence of (l,σ,v). Let,

$$\begin{aligned} v'(l,\sigma,v) &= \max_{\sigma_0 \in \Delta\{in,out\}} \sigma_0 v'_b(in,l,\sigma,v) + (1-\sigma_0)h_{bG(b)} \\ S'_{b,0}(l,\sigma,v) &= \arg\max_{\sigma_0 \in \Delta\{in,out\}} \sigma_0 v'_b(in,l,\sigma,v) + (1-\sigma_0)h_{bG(b)}. \end{aligned}$$

Again, by the same reasoning as above, since $v'_b(in,l,\sigma,v)$ is continuous, $v'(l,\sigma,v)$ is continuous, $S'_{b,0}(l,\sigma,v)$ is UHC, nonempty, convex and compact valued.

Similarly, for a seller, let

$$\begin{aligned} v'_s(in|l,\sigma,v) = \max\nolimits_{m'_s,t'_s} -c + (1-\beta)\sum\nolimits_{b\in B} p_b(t'_{sb} - m'_{sb}(r_s + \delta v_s)) \\ + \beta \sum\nolimits_{b\in B} p_b(t_{bs} + m_{bs}(r_s + \delta v_s)) + \delta v_s \end{aligned}$$

subject to

$$\begin{aligned} m'_{sb}(\delta v_{b\cup s} - \delta v_b) - t'_{sb} &\geq m'_{sj}(\delta v_{b_\cup s} - \delta v_b) - t'_{sj} \text{ for all } b \text{ and } j \in B \\ m'_{sb}(\delta v_{b\cup s} - \delta v_b) - t'_{sb} &\geq 0 \text{ for all } b \\ m'_{sb} &\leq 1 \text{ for all } b \end{aligned}$$

Also, let $S'_{s,1}(l,\sigma,v)$ denote the set of maximizers for the above program and

$$\begin{aligned} v'_s(l,\sigma,v) &= \max_{\sigma_0 \in \Delta\{in,out\}} \sigma_0 v'_s(in,l,\sigma,v) \\ S'_{s,0}(l,\sigma,v) &= \arg\max_{\sigma_0 \in \Delta\{in,out\}} \sigma_0 v'_s(in,l,\sigma,v). \end{aligned}$$

Finally let $S'_i(l,\sigma,v) = S'_{i,0}(l,\sigma,v) \times S'_{s,1}(l,\sigma,v)$ and $S'(l,\sigma,v) = \prod_i S'_i(l,\sigma,v)$. Define correspondence $(l,\sigma,v) \mapsto (l', S', v')$. This correspondence maps $\Lambda \times \Sigma \times V$ into $\Lambda \times \Sigma \times V$, is UHC, compact, and convex valued; thus by Kakutani's theorem has a fixed point. This fixed point is an equilibrium for the economy. $\square$

References

DeFraja, G., and J. Sakovics (2001): "Walras Retrouve: Decentralized Trading Mechanisms and the Competitive Price," *Journal of Political Economy*, 109, 842–863.

Gale, D. (1987): "Limit Theorems for Markets with Sequential Bargaining," *Journal of Economic Theory*, 43, 20–54.

Gul, F., and E. Stacchetti (1999): "Walrasian Equilibrium with Gross Substitutes," *Journal of Economic Theory*.

——— (2000): "The English Auction with Differentiated Commodities," *Journal of Economic Theory*, 92, 66 – 95.

Kelso, A., and V. Crawford (1982): "Job Matching, Coalition Formation, and Gross Sustitutes," *Econometrica*, 50(6), 1483–1504.

Lauermann, S. (2006): "Dynamic Matching and Bargaining Games: A General Approach," *Mimeo.*

——— (2008): "Price Setting in a Decentralized Market and the Competitive Outcome," *Mimeo.*

Maskin, E., and J. Tirole (1990): "The Principal-Agent Relationship with an Informed Principal: The Case of Private Values," *Econometrica*, 58(2), 379–409.

Mortensen, D., and R. Wright (2002): "Competitive Pricing and Efficiency in Search Equilibrium," *International Economic Review*, 43(1), 1–20.

Myerson, R. (1983): "Mechanism Design by an Informed Principal," *Econometrica*, 51(6), 1767–1798.

Riley, J., and R. Zeckhauser (1983): "Optimal Selling Strategies: When to Haggle, When to Hold Firm," *Quarterly Journal of Economics*, 98(2), 267–289.

Roth, A., and M. Sotomayor (1990): *Two Sided Matching: A Study in Game-Theoretic Modelling and Analysis*. Cambridge University Press, Cambridge, UK.

Satterthwaite, M., and A. Shneyerov (2007): "Dynamic Matching, Two-sided Incomplete Information, and Participation Costs: Existence and Convergence to Perfect Competition," *Econometrica*, 75(1), 155–200.

——— (2008): "Convergence to Perfect Competition of a Dynamic Matching and Bargaining Market with Two-sided Incomplete Information and Exogenous Exit Rate," *Mimeo*.

Serrano, R. (2002): "Decentralized Information and the Walrasian Outcome: A Pairwise Meetings Market with Private Values," *Journal of Mathematical Economics*, 38, 65–89.

Shapley, L., and M. Shubik (1972): "The Assignment Game I: The Core," *International Journal of Game Theory*, 1(2), 111–130.

Shneyerov, A., and A. Wong (2007): "The Rate of Convergence to Perfect Competition of a Simple Matching and Bargaining Mechanism," *Mimeo*.

Wolinsky, A. (1990): "Information Revelation in a Market with Pairwise Meetings," *Econometrica*, 58, 1–23.

Yilankaya, O. (1999): "A Note on the Seller's Optimal Mechanism in Bilateral Trade with Two-Sided Incomplete Information," *Journal of Economic Theory*, 87, 267–271.

MEDS, Kellogg School of Management, Northwestern University, 2001 Sheridan Road, Evanston, IL 60208

E-mail address: a-atakan@kellogg.northwestern.edu