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Alp E. Atakan

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Rumeli Feneri Yolu 34450 Sarıyer/Istanbul

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ABSTRACT. This paper considers a frictional market where buyers and sellers, with unit demand and supply, search for trading opportunities. The analysis focuses on explicit search frictions, allows for two-sided incomplete information, and puts no restriction on agent heterogeneity. In this context, a non-trivial, full trade search equilibrium is shown to exist, equilibria are characterized as the values that satisfy the first order conditions for a non-linear planner's (optimization) problem, and necessary and sufficient conditions are provided for the existence of efficient search equilibria under complete information. These results fully generalize to the two-sided incomplete information setting, under an additive separability condition.

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1. INTRODUCTION

On an online marketplace, such as Ebay (or maybe even Google), sellers endowed with heterogeneous objects, search for potential buyers by running auctions or posting prices, in turn, buyers look for objects by browsing through listings. In the labor market individuals who differ in various dimensions try to locate jobs that complement their skills and suit their tastes. Families sort through housing options to locate a match that fulfils their diverse needs. These markets share a common structure: heterogeneous sellers try to find an appropriate trading partner from a diverse set of potential buyers. The Assignment Problem (see Shapley and Shubik (1972), or Roth and Sotomayor (1990) for a more recent treatment) is the canonical model for analyzing such markets for indivisible objects, when trade occurs in a centralized, frictionless competitive market. However, finding a partner is most commonly a decentralized process that is (almost) never frictionless. Participating and remaining

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active in a marketplace, and then locating and verifying the attributes of a potential match partner, involves monetary and non-monetary (hassle) costs; these are the explicit costs of search. The possibly large amount of time spent finding a match partner is also an implicit cost of search. Moreover, agent's frequently have private information about the traded goods as well as their preferences. Asymmetric information further exacerbates both implicit and explicit search costs since even buyers and seller pair, with large gains from trade, may fail to trade.

There is a large body of work that uses decentralized matching and bargaining models to analyze frictional exchange. Much research has focused on characterizing and outlining the efficiency properties of equilibria in decentralized matching and bargaining models with search frictions. (For example, Mortensen (1982), and Hosios (1990) explore efficiency properties of markets with homogeneous buyers and sellers; Sattinger (1995) focuses on the multiplicity of equilibria in a model with discounting and heterogeneous agents; Shimer and Smith (2000) establishes existence and characterize matching patterns while Shimer and Smith (2001) explores efficiency properties for models with discounting and heterogenous agents). Given that inefficiencies may exits with frictions, researchers have also addressed whether these models converge to an efficient, competitive market, as search frictions become small (For a homogeneous good Rubinstein and Wolinsky (1985), Gale (1987) and Mortensen and Wright (2002) explore convergence to a competitive equilibrium under complete information while Satterthwaite and Shneyerov (2007) extend the analysis to the two-sided incomplete information case). However, the bulk of the previous literature focuses on implicit time costs due to discounting rather than explicit costs of search and assumes complete information (see Atakan (2006) and Chade (2001), for two exceptions, that characterize matching patterns with explicit search frictions). Also, with a few exceptions, the models lack the full richness of the Assignment Problem in terms of buyer and seller heterogeneity.

The analysis here focuses on explicit search frictions, allows for two-sided incomplete information, and puts no restriction on agent heterogeneity. Search frictions are modeled as an additive per-period cost of search; locating a trading partner is "fast," but nevertheless costly. For example, in an online marketplace an agent may locate a potential trade almost immediately, but must pay a listing/sampling fee, or alternatively, must maintain the required technology for participating in the marketplace. There are no restrictions on preferences or endowments, except transferable utility and unit demand/supply. Consequently, there is heterogeneity in buyer tastes, seller costs and the good each sellers owns - the model is the frictional analogue of the Assignment Problem. In this context, a non-trivial, full trade search equilibrium is shown to exist. Equilibria are characterized as the set of values that satisfy the first order conditions for a non-linear planner's (optimization) problem, and necessary and sufficient conditions are provided for the existence of efficient search equilibria under complete information. Under an additive separability assumption, these results generalize to the two-sided incomplete information setting where incomplete information is modeled as a case of independent private values. Also, search equilibria are shown to converge to a competitive equilibria under both complete and incomplete information. However, inefficiencies may remain at the limit even under complete information: the limiting competitive equilibrium can be an equilibrium for only a proper subset of the full economy under consideration.

A description of the model is as follows: In each period a unit measure of each type (of buyers and sellers) from a finite set of types is available for entry and those who expect a positive payoff voluntarily enter the market. The market is in steady-state with the measure of agent types endogenously determined to balance the flow of types through the economy. Once in the market, each agent pays a per period cost, and receives a "draw" from the distribution of active players. The probability that any buyer (or seller) is paired with a particular type is proportional to the frequency of that type among all sellers (buyers) active in steady state. After two agents are paired, nature designates a proposer, the proposer makes a price offer, and the responder decides whether to trade at the offered price. If a meeting between a pair results in a trade, then the two agents are removed from the population, otherwise the agents return to the population of active players. Efficiency in this context is defined by means of a planner's problem. The planner also maintains a steady state and is constrained by the decentralized search technology for creating buyer-seller pairs. However, the planner can choose the measure of agents in the market by controlling entry decisions and can choose which types consummate a match if paired, i.e., match probabilities. Consequently, the planner maximizes per-period production net of search costs by choosing the match probabilities and the steady state measure subject to the market remaining in steady state. If there are no search costs, then the planner's problem is equivalent to the Assignment Problem.

Search equilibria are characterized using a hypothetical planner's problem which may differ from the actual planner's problem only in terms of distorted buyer and seller search costs. The central result of the paper shows, under complete information or two-sided incomplete information satisfying an additive separability condition, the set of search equilibrium match probabilities and steady state measures coincides with those that satisfy the first order conditions for the hypothetical planner's problem. In the model, search costs and the likelihood of proposing jointly determine the bargaining strength of a buyer vis-à-vis a seller, or alternatively, agents' perception of their search costs are distorted by their relative bargaining strength. If buyers and sellers are symmetric in their bargaining strength, then search costs are undistorted, the hypothetical problem coincides with the actual planner's problem and an efficient search equilibrium exists.[1] If buyers and sellers differ in their bargaining strengths on the other hand, then the equilibrium is inefficient: there is excess entry by the relatively strong side of the market and insufficient entry by the weak side in every equilibrium with trade. Even when an efficient equilibrium exists, there are also inefficient search equilibria. The planner's problem is not convex and so search equilibria that satisfy the first order conditions may nevertheless fail to be optimal. This kind of inefficiency stems from a lack of coordination between buyers and sellers in their entry decisions. A trivial equilibrium where nobody enters the market always exists and provides a stark demonstration of a failure to coordinate entry. More generally, if agents are symmetric in their bargaining strength, then any inefficiency that may remain is shown to stem from coordination failures.

As search costs become small the welfare cost of inefficient entry and any inefficiency due to asymmetric information disappears as the market converges to a competitive equilibrium. Inefficiency due to coordination failures may remain, however, even at the limit. Section 3.3 provides a “stable” example of a coordination failure that results in a missing market for any level of search costs and consequently at the limit.[2] The limit in this example is a competitive equilibrium with trade for a proper

[1]In the model, equal bargaining power implies that the Hosios' Condition (see Mortensen (1982) or Hosios (1990)) is satisfied in equilibrium.

[2]Observe that the trivial no-trade equilibrium mention above also exists for any level of search costs and the limit is a competitive equilibrium for the empty market. This example is, however, fragile,

subset of the original economy. Nevertheless, the "best" search equilibrium of the model, i.e., the equilibrium, free of coordination failures, that solves the hypothetical planner's problem, converges to the competitive equilibrium of the original economy and is therefore asymptotically efficient. The competitive equilibrium benchmark under consideration is a "flow" equilibrium as in Gale (1987) or Satterthwaite and Shneyerov (2007), generalized to accommodate heterogenous goods. In each period, flow supply is the measure of sellers of a particular good entering the market and flow demand is the measure of agents willing to purchase a particular good entering the market. In a flow equilibrium, the buyer and seller continuation values, which are the implicit prices, equate flow supply to flow demand for each of the goods traded in the market.

The paper proceeds as follows: Section 2 outlines the decentralized economy as well as the planner's problem, Section 3.1 presents the main results for the complete information version of the model, Section 3.2 extends the result to the incomplete information setting, Section 3.3 shows convergence to competitive equilibria, Section 3.4 discusses some extensions and Section 4 concludes. Proofs that are not included in the main text are in the Appendix.

2. The Model

Buyers and sellers in the economy engage in search for possible trading partners. Each seller owns one indivisible good for sale and each buyer wants to buy one good. Time is taken as discrete. In each period, agents incur a positive, search cost $c_B \geq 0$ for a buyer and $c_S \geq 0$ for a seller and meet pairwise with potential partners. Either the buyer or the seller is designated as the proposer. The probability that the buyer is designated as the proposer is $\beta \in (0, 1)$. The proposer offers a price. If the responder accepts, then trade takes place and both agents leave the market. Agents who fail to trade in a period return to the population and continue searching for potential match partners. Utility is transferable. If a buyer of type b and a seller of type s consummate their match, then they create total utility f_{bs} which they split between themselves.

i.e., not "stable," since if some buyer accidently were to enter the market then some sellers would follow and the trivial equilibrium would unravel.

2.1. **Population of Agents.** B and S denote the finite sets of buyer and seller types and $I = B \cup S$ the set of all types. In each period, a unit measure of each type considers entering the market. Assume that the market is in steady state and let $l = \left(l_{b_1}, ..., l_{|B|}, l_{s_1}, ..., l_{|S|}\right)$ denote the steady state measure of buyers and sellers in the market, i.e., $l \in \mathbb{R}_+^{|B|+|S|}$. The probability for any seller of meeting buyer b in a given period is $p_b = \alpha l_b / \max\{L_B, L_S\}$, where $L_B = \sum_{b \in B} l_b$ and $\alpha \in (0, 1]$, for expositional simplicity only take $\alpha = 1$.[3] Likewise, the probability of meeting seller s is $p_s = l_s / \max\{L_B, L_S\}$. In the case where $l_i = 0$ for all $i \in I$, the total measure of pairs formed is equal to zero. These probabilities are commonly known by all agents. In the incomplete information set-up of Section 3.2, further assume any agent's prior belief about his/her trading partner's type coincides with the steady state probabilities.

2.2. **Agent Behavior and Strategies.** Let σ_i denote a strategy for player i and $\sigma = (\sigma_i)_{i \in I}$ a strategy profile. In the first period for agent b, the strategy determines a probability of entering the market. Given that agent b is paired with s in the current period, σ_b determines the price offer t, made to agent s, if agent b is designated as the proposer and a probability of accepting price offer t, made by agent s, if b is designated as the responder.[4] The per-period reward function for a buyer b (seller s) paired in the current period with seller s (buyer b) is:

$$\pi_b(\sigma, s) \text{ (or } \pi_s(\sigma, b)) = \begin{cases} -c_B + f_{bs} - t_{bs}(\sigma) \\ -c_B + f_{bs} - t_{sb}(\sigma) \\ -c_B \end{cases} \left(\begin{array}{cc} & -c_S + t_{bs}(\sigma) \\ \text{or} & -c_S + t_{sb}(\sigma) \\ & -c_S \end{array} \right) \begin{array}{l} \text{proposal of } b \text{ accepted,} \\ \text{proposal of } s \text{ accepted,} \\ \text{proposal rejected,} \end{array}$$

where $t_{bs}(\sigma)$ denotes the price offer made by buyer b to seller s given strategy σ. If an agent does not enter the market or has accepted a match in a prior period, then the agent is paired with 0, and $\pi_i(\sigma, 0) = 0$. Also, if the agent does not get paired in a period, then she/he is paired with herself and $\pi_i(\sigma, i) = -c_i$. Buyers and sellers

[3]This assumption is without loss of generality and it is easily verified that all results presented in the paper go through for any choice of α.

[4]The initial probability of entry and the probability of accepting an offer for a certain type is interpreted as the corresponding proportion of agents of that type playing a pure strategy.

solve the following problem after any history of the game:

$$v_i = \max_{\sigma_i} \left[\mathbb{E} \sum_{t=0}^{\infty} \pi_i \left(\sigma_i, \sigma, j\left(t\right) \right) \right],$$

where $j\left(t\right)$ is drawn according to the steady state measure l, if there was a rejection in period $t-1$. Since stationary sub-game perfect equilibria are considered, strategies must maximize the infinite sum of payoffs after any possible path of play.

2.3. **Steady State.** Given a strategy profile σ, let $m_b \leq 1$ denote the number of buyers of type b entering the market in each period. Also, let m_{bs} denote the probability that b and s match and leave the market, given that they are paired in a period and b is the proposer. The market is assumed in steady state and so:

(SS for Buyers) $$l_b\beta \sum\nolimits_s p_s m_{bs} + l_b(1-\beta) \sum\nolimits_s p_s m_{sb} = m_b \leq 1 \; \forall b \text{ and,}$$

(SS for Sellers) $$l_s\beta \sum\nolimits_b p_b m_{bs} + l_s(1-\beta) \sum\nolimits_b p_b m_{sb} = m_s \leq 1 \; \forall s.$$

These equations state that the number of type b buyers (or type s sellers) entering the market in each period must equal the number of that type leaving the market. Note that an agent will enter the market and search only if their value from participating, i.e., v_i, is non-negative. Also, if $v_i > 0$, then all type i agents (equal to measure one) will choose to participate each period.

2.4. **Search Equilibrium.** A search equilibrium is comprised of a mutually compatible strategy profile σ and steady state measure l, that is, the measure l satisfies the steady state equations, given that agents use strategy profile σ and, each σ_i is optimal after any sub-game given that agents use σ and the steady state measure is l.

Given the stationary environment, values $v = (v_b, v_s)_{b \in B, s \in S}$ are also option values of remaining unmatched in the economy. In any search equilibrium, a proposer will offer no more than the continuation payoff to a responder. So, a buyer offers seller s no more than v_s and a seller offers buyer b no more than $f_{bs} - v_b$. Consequently, values satisfy the recursive equations

$$v_b = \max\{-c_B + \sum\nolimits_s \beta p_s s_{bs}^{+} + v_b, 0\},$$
$$v_s = \max\{-c_S + \sum\nolimits_b \left(1-\beta\right) p_b s_{bs}^{+} + v_s, 0\},$$

where $s_{bs} = f_{bs} - v_b - v_s$ denotes the surplus created in a match between b and s and $s_{bs}^+ = \max\{s_{bs}, 0\}$. In subsequent analysis, it is more convenient to deal with the values v, matching probabilities m and the steady state measure l instead of detailing the equilibrium strategy profile σ. For notational convenience only, assume that $m_{bs} = m_{sb}$ unless otherwise stated.[5] The following four conditions are met by any l, m, and v in a search equilibrium.

1. Individual Rationality. *$v_i \geq 0$ for all i.*

2. Efficient Bargaining. *If $s_{bs} > 0$, then $m_{bs} = m_{sb} = 1$ and if $s_{bs} < 0$, then $m_{bs} = m_{sb} = 0$.*

3. Constant Surplus. *The surplus function s_{bs} satisfies*

$$\beta \sum\nolimits_s p_s m_{bs} s_{bs} = \beta \sum\nolimits_s p_s s_{bs}^+ \leq c_B \text{ for all } b$$
$$(1-\beta) \sum\nolimits_b p_b m_{bs} s_{bs} = (1-\beta) \sum\nolimits_b p_b s_{bs}^+ \leq c_S \text{ for all } s$$

where the inequality holds with equality for i with $p_i > 0$.

4. Steady State. *$l_i \sum_j m_{ij} p_j \leq 1$ and if $v_i > 0$, then the inequality for i holds with equality.*

The Individual Rationality Condition holds since entry to the market is voluntary. The Efficient Bargaining Condition follows since in a random proposer game, any meeting between b and s with positive surplus, results in a certain match. The Constant Surplus Condition is a restatement of the recursive equations for buyer and seller values. Finally, the Steady State Condition follows since if the value from entering the economy for a certain type is strictly positive, then all potential entrants of that type must enter the market.

As argued above if l and σ form a search equilibrium, then the implied v, l and m satisfy Conditions 1 through 4. The following proposition gives the converse and demonstrates that restricting attention to (v, l, m) triples is without loss of generality.

[5]This symmetry assumption only has bite if the agents are indifferent between accepting each other. In this case, if $m_{bs} \neq m_{sb}$, then define a symmetric equilibrium with $\hat{m}_{bs} = \hat{m}_{sb} = \beta m_{bs} + (1-\beta) m_{sb}$.

Proposition 1. *If l, m and v satisfy Conditions 1 through 4, then there exists a search equilibrium l and σ where m and v are the equilibrium match probabilities and values.*

2.5. **The Planner's Problem.** As a benchmark for efficiency, a planner's problem is considered. The planner maximizes the per-period total production in the steady state economy. However, the planner is constrained by the same technology as the decentralized market in the formation of buyer seller pairs. The planner controls agents' entry, exit and matching decisions. Consequently, the planner chooses the steady state measure and the match probabilities, subject to ensuring that the market remains in steady state. The planner's problem is

$$W(I, c_B, c_S) = \max_{l,m} W(I, c_B, c_S, l_b, l_s, m) = \max_{l,m} \frac{\sum_b \sum_s l_b l_s m_{bs} f_{bs}}{\max\{L_B, L_S\}} - c_B L_B - c_S L_S \tag{1}$$

subject to

$$l_b \sum\nolimits_s \frac{l_s}{\max\{L_B, L_S\}} m_{bs} \leq 1 \text{ for all } b \qquad (v_b) \tag{2}$$

$$l_s \sum\nolimits_b \frac{l_b}{\max\{L_B, L_S\}} m_{bs} \leq 1 \text{ for all } s \qquad (v_s) \tag{3}$$

$$0 \leq m_{bs} \leq 1 \text{ for all } b \text{ and } s \qquad (\chi_{bs} \text{ and } \omega_{bs}) \tag{4}$$

$$l_i \geq 0 \text{ for all } i \in I = B \cup S, \qquad (\mu_i) \tag{5}$$

where the associated Kuhn-Tucker multiplier for each constraint is given in parenthesis to the right. With a slight abuse of notation, I summarizes the whole economy, i.e., denotes the set of agents as well as the joint production function f. Also, let $W(I, c_B, c_S, l_s) = \max_{\hat{l}_b, m} W(I, c_B, c_S, l_b, l_s, m)$ denote the planner's problem where the measure of sellers l_s is exogenously given. The objective function, given in equation (1), is the total production per period net of search costs. Total production in a period equals the measure of b and s matches, $l_b l_s m_{bs} / \max\{L_B, L_S\}$, times production in this match f_{bs}, summed over all type pairs, while aggregate cost of search in a period equals $c_B L_B + c_S L_S$. Constraints given by equation (2) and (3) ensure that the number of agents of a certain type leaving the market as a result of a successful match is less than one. The inequality need not hold with equality since the planner can choose to have fewer than the maximum number of a particular type enter the market. The constraint given by equation (4) ensures that the match probabilities lie between zero and one. Finally, equation (5) is the non-negativity constraint for the steady state measure. If $l_i = 0$ for all i, then the above problem is not well defined. In this case, the total number of matches formed is assumed to equal zero; thus all constraints are satisfied and the objective function is equal to zero.

The measure of buyers and sellers in the market must be equal in any solution to the planner's problem (Lemma 1 below). Since if $L_B > L_S$, then L_B can be scaled down without affecting the distribution of types, and thus keeping the number of pairs created constant while strictly decreasing the search costs. Also, using Lemma 1 to impose $L = L_B = L_S$ when solving the planner's problem ensures differentiability and simplifies the analysis.

Lemma 1. *If $c_B > 0$ and $c_S > 0$, then in any solution to the planner's problem $L_B = L_S$.*

For $c_B > 0$ or $c_S > 0$, it is straight forward to show that a solution to the planner's problem exists. The following proposition further shows that when $c_B = 0$ and $c_S = 0$, the planner's problem is equivalent to the Assignment Problem where fractional assignments are permitted and consequently has a solution. The Assignment Problem is the competitive benchmark for the economy under consideration and characterizes all the "flow" competitive (and Pareto optimal) allocations for the economy.

Proposition 2. *For all $c_B \geq 0$ and $c_S \geq 0$ a solution to the planner's problem exists.*

3. Main Results

3.1. **Existence of Efficient Equilibria.** The planner's problem provides a convenient tool for proving existence and characterizing all search equilibria. Assume that $c_B > 0$, $c_S > 0$ and without loss of generality,

$$r = \frac{c_S/(1-\beta)}{c_B/\beta} \geq 1.$$

Consider a hypothetical planner's problem $W(I, c_S/2(1-\beta), c_S/2(1-\beta), l, m)$, i.e., a problem where the per period search cost for a buyer and a seller is $c_S/2(1-\beta)$, instead of c_B and c_S. Let $\mathcal{F}$ denote the set of match probabilities m, steady state measures l and Kuhn-Tucker multipliers v_b and v_s, associated with equations (2) and (3), that satisfy the first order conditions for $W(I, c_S/2(1-\beta), c_S/2(1-\beta), l, m)$.

Theorem 1 below shows that the set of search equilibria l, m and v, denoted $\mathcal{E}$, are just the elements of $\mathcal{F}$, but with the measure of buyers l_b multiplied by r. The measure of buyers is increased by r since in a search equilibria the measure of sellers and buyers in the market are not equal (Lemma 2 below), whereas in any solution to the planner's problem $L_B = L_S$ (Lemma 1).

Lemma 2. *If $L_S > 0$, then $L_B/L_S = r$.*

Proof. For $l_b > 0$, multiplying both sides of the constant surplus condition by l_b implies that

$$l_b c_B = \beta \sum\nolimits_s l_b p_s m_{bs} s_{bs}. \tag{6}$$

Summing over all buyers shows that

$$L_B c_B = \beta \sum\nolimits_{B\times S} l_b p_s m_{bs} s_{bs}. \tag{7}$$

Repeating the same steps for the sellers implies that

$$L_S c_S = (1-\beta) \sum\nolimits_{B\times S} l_s p_b m_{bs} s_{bs}. \tag{8}$$

However, $l_s p_b = l_b p_s$ for all b and s, so $L_B C_B/\beta = L_S C_S/(1-\beta)$ and $L_B/L_S = r$. □

Theorem 1 characterizes all search equilibria as the set of (l, m, v) that satisfy first order conditions for a non-linear program (the hypothetical planner's problem). The theorem also shows that a full-trade, non-trivial search equilibrium exists by proving that the set $\mathcal{F}$ includes an optimizer of the planner's problem. The planner's problem has an optimizer, by Proposition 2. Theorem 1 further shows the existence of Kuhn-Tucker multipliers at any optimizer of the planner's problem by proving that a certain constraint qualification holds.[6]

Theorem 1. *The set of search equilibria $\mathcal{E} = \{(l, m, v) : (l, m, v) \in \mathcal{F}\}$ and $\mathcal{E} \neq \varnothing$.*

This theorem is stated under the assumption that $c_b > 0$ and $c_S > 0$. If $c_B = c_S = 0$, then r is not defined and Lemma 1 is no longer valid, however Theorem 2 is still true. In particular, if $c_B = c_S = 0$, then $\mathcal{E} = \mathcal{F}$, which follows immediately from the proof provided for Theorem 1 in the appendix. This point is further elaborated in Section 3.3 where limiting equilibria and equilibria at the limit are discussed.

[6]Observe that the standard constraint qualifications do not work since the planner's problem is not a convex optimization problem

In general, the problem set-up to characterize search equilibria and the actual planner's problem are not the same, thus search equilibria are not necessarily efficient. However, if $c_B/\beta = c_S/(1-\beta)$, i.e., if $c_B + c_S = c_S/(1-\beta)$, then the two problems coincide and an efficient search equilibrium exists (Corollary 1 below). The bargaining strength of an agent is jointly determined by the probability of proposing β, and search cost c. Hence, efficiency requires that agents have equal bargaining power. Symmetry in terms of bargaining power, i.e., $c_B/\beta = c_S/(1-\beta)$, implies by the constant surplus condition, that agents share equally the expected surplus from future matches. The following corollary shows that equal bargaining power, or the symmetric division of the surplus between buyers and sellers, is a sufficient condition for an efficient search equilibrium. Also, if efficient matching involves trade, i.e., $L_S > 0$ in the planner's problem, then $c_B/\beta = c_S/(1-\beta)$ is also a necessary condition for the existence of an efficient search equilibrium. This is because if $c_B/\beta < c_S/(1-\beta)$, in any search equilibrium with trade, $L_B > L_S > 0$, by Lemma 2, i.e., there is excess entry by buyers.

Corollary 1. *If $c_B/\beta = c_S/(1-\beta)$, then there exists an efficient search equilibrium (l, m, v), and $W(c_B, c_S) = \sum_b v_b + \sum_s v_s = \max_{(\hat{l},\hat{m},\hat{v})\in\mathcal{E}} \sum_b \hat{v}_b + \sum_s \hat{v}_s$.*

Proof. If $(l, m) \in \arg\max W(c_B, c_S, l, m)$, then $(l, m, v) \in \mathcal{E}$ by Theorem 1 and since $c_S/(1-\beta) = c_B + c_S$. Also, rearranging equation (8), by recalling that $s_{bs} = f_{bs} - v_s - v_b$ and that for $v_i > 0$, $\sum_j l_i p_j m_{ij} = 1$, shows

$$\sum_s v_s + \sum_b v_b = -c_B L_B - c_S L_S + \sum_{B\times S} l_b p_s m_{bs} f_{bs} \tag{9}$$

for all $(l, m, v) \in \mathcal{E}$. Consequently,

$$\max_{(\hat{l},\hat{m},\hat{v})\in\mathcal{E}} \sum_b \hat{v}_b + \sum_s \hat{v}_s = \max_{l,m} W(c_B, c_S, l, m). \tag{10}$$

□

Even if the sufficient condition given in Corollary 1 is satisfied, there are inefficient search equilibria in addition to the efficient equilibria identified above. The planner's problem is not a convex optimization, so a search equilibria that, by Theorem 1, satisfies the first order conditions may not be an optimizer of the planner's problem. This kind of inefficiency can be related to coordination problems. A trivial equilibrium where nobody enters the market always exists and provides a stark demonstration of a failure between buyers and sellers to coordinate. A two buyer, two seller search market provides a more robust failure of coordination. Let $f_{12} = f_{21} = 1/5$ and $f_{11} = f_{22} = 1$, that is buyer 1 likes seller 1's good and buyer 2 likes seller 2's good. Assume that $c_B = c_S = c$ and $\beta = 1/2$. The efficient equilibrium involves type 1 buyers trading only with type 1 sellers, and type 2's trading only with other type 2's for small c, and for $3/10 \geq c \geq 2/10$ requires all buyers trading with all sellers. It is straight forward to show that there are stable inefficient equilibria for all $c \leq 3/10$, where only type 1 (or only type 2) agents enter, i.e., there is a coordination failure between sellers and buyers of type 2. These inefficient equilibria remain, and remain stable, even as search costs disappear. This point is further developed in section 3.3.

The following corollary proves that if $c_B/\beta = c_S/(1-\beta)$, then any inefficiency in a search equilibrium is caused by coordination failures. More precisely, if the measure of sellers l_s (or buyers l_b) is taken to equal it's search equilibrium value, then the equilibrium measure of buyers and the equilibrium match probability is efficient. That is, the equilibrium measure of buyers, as well as the matching is efficient, when the measure of sellers is taken as exogenous. Consequently, if sellers and buyers are symmetric in their bargaining strength, then the only source of inefficiency is agents not coordinating with the opposite side of the market in their market entry decisions. In other words, equilibrium option values are sufficient to coordinate matching decisions for all agents and entry/exit decisions for all buyers (or for all sellers), but not sufficient to jointly coordinate entry decisions for both sides of the market.

Corollary 2. *Assume $c_B/\beta = c_S/(1-\beta)$. If l, m and v comprise a search equilibrium, then*

(i) The steady state measure of buyers l_b and the matching m is optimal given the measure of seller l_s in the market, i.e., $(l_b, m) \in \arg\max_{l_b,m} W(c_B, c_S, l_b, l_s, m)$,
(ii) The steady state measure of sellers l_s and the matching m is optimal given the measure of buyers l_b in the market, i.e., $(l_s, m) \in \arg\max_{l_s,m} W(c_B, c_S, l_b, l_s, m)$.

Lemma 2 shows that whenever the surplus is shared unevenly between buyers and seller, that is whenever $c_B/\beta \neq c_S/(1-\beta)$, there is inefficient entry into the market, even if the search equilibrium is an optimizer to the hypothetical planner's problem. A two type example of a market for a homogeneous good is sufficient to further demonstrate inefficiency due to excess or insufficient entry. Consider an economy where high (b_h) and low (b_l) valuation buyers are willing to pay 1 and $1/2-\epsilon$ respectively for a good, while high (s_h) and low (s_l) cost sellers can produce the good for $1/2+\epsilon$ and 0, respectively. Assume $c_B+c_S \leq 1/2$ and $\beta = 1/2$. Efficiency requires that only types b_h and s_l enter in each period, an equal number of buyers and sellers are present in the market (Lemma 1), and each meeting ends in a trade. Under these parameter values, there are two search equilibria: a trivial equilibria without trade, and an equilibrium where only b_h and s_l enter and all meetings end in a trade. By Lemma 2, $l_{b_h}/l_{s_l} = c_S/c_B$ and so the equilibrium is efficient only if the surplus is shared equally between buyers and sellers, i.e., if $c_B = c_S$. Since otherwise, e.g., if $c_S > c_B$, then $l_{s_l} < l_{b_h}$ and there is *excess entry* by types b_h. Instead if we assume, $c_B < 1/2 < c_S$, then no trade is the only search equilibrium, but for $c_B + c_S \leq 1$ the efficient configuration is as before. In this search equilibrium, sellers do not receive enough surplus with $\beta = 1/2$ and do not enter the market; thus there is *insufficient entry* by types s_l.

The intuition of the two type example is extended to the general model in the following corollary. Suppose that $c_S/1-\beta > c_B/\beta$, i.e., buyers are in a stronger bargaining position and receive more of the surplus. The corollary shows the existence of a "best" search equilibrium that optimizes a hypothetical planner's problem with search costs equal to $c_S/(1-\beta)$, if the measure of buyers L_B is scaled down by r. This implies that there are "too few" sellers in the market when compared to actual planner's problem since $c_S/(1-\beta) > c_B + c_S$ and there are "too many" buyers in the market given the measure of sellers is L_S, since $L_B > L_S$. Also, the matching and distribution of types solve the hypothetical planner's problem, consequently, excess entry by buyers and insufficient entry by sellers are the only sources of inefficiency in the best search equilibrium.

Corollary 3. *There exists* $(l, m, v) \in \mathcal{E}$ *such that*

$$(l_b/r, l_s, m) \in \arg\max W\left(c_S/2\left(1-\beta\right), c_S/2\left(1-\beta\right), l, m\right),$$

and (l, m, v) *is the best search equilibrium, i.e.,* $\sum_b v_b + \sum_s v_s = \max_{\hat{v}\in\mathcal{E}} \sum_b \hat{v}_b + \sum_s \hat{v}_s$. *Also,* $L_S \leq L'_S$ *for any* $(l', m') \in \arg\max W(c_B, c_S, l, m)$.

Proof. If $(l_b/r, l_s, m) \in \arg\max W\left(c_S/2\left(1-\beta\right), c_S/2\left(1-\beta\right), l, m\right)$, then $(l_b/r, l_s, m, v) \in \mathcal{F}$ for some v and $(l_b, l_s, m, v) \in \mathcal{E}$, by Theorem 1. Also, by the argument given in Corollary 1, $\sum \hat{v}_b + \sum \hat{v}_s = -L_S c_S/(1-\beta) + \sum_{B\times S} \hat{l}_s \hat{p}_b \hat{m}_{bs} f_{bs}$, for all $(\hat{l}, \hat{m}, \hat{v}) \in \mathcal{E}$.

Consequently, $\sum_b v_b + \sum_s v_s = \max_{\hat{v}\in\mathcal{E}} \sum_b \hat{v}_b + \sum_s \hat{v}_s$. To show $L_S \leq L'_S$ assume to the contrary that $L_S > L'_S$ for some $(l', m') \in \arg\max W(c_B, c_S, l, m)$. Optimality of $(l_b/r, l_s, m)$ for cost $c_S/2(1-\beta)$ implies that

$$-L_S \frac{c_S}{1-\beta} + \sum\nolimits_{B\times S} l_s p_b m_{bs} f_{bs} \geq -L'_S \frac{c_S}{1-\beta} + \sum\nolimits_{B\times S} l'_s p'_b m'_{bs} f_{bs} \tag{11}$$

$$\sum\nolimits_{B\times S} l_s p_b m_{bs} f_{bs} - \sum\nolimits_{B\times S} l'_s p'_b m'_{bs} f_{bs} \geq (L_S - L'_S)\frac{c_S}{1-\beta} \tag{12}$$

However,

$$(L_S - L'_S)\frac{c_S}{1-\beta} > (L_S - L'_S)(c_S + c_B) \tag{13}$$

and so

$$\sum\nolimits_{B\times S} l_s p_b m_{bs} f_{bs} - \sum\nolimits_{B\times S} l'_s p'_b m'_{bs} f_{bs} > (L_S - L'_S)(c_S + c_B) \tag{14}$$

$$-L_S(c_S + c_B) + \sum\nolimits_{B\times S} l_s p_b m_{bs} f_{bs} > -L'_S(c_S + c_B) + \sum\nolimits_{B\times S} l'_s p'_b m'_{bs} f_{bs} \tag{15}$$

contradicting that $(l', m') \in \arg\max W(c_B, c_S, l, m)$. $\square$

3.2. **Efficiency under Incomplete Information.** Here, the results presented in the previous subsection are extended to the case where buyers and sellers in the market have private information under certain conditions. In the incomplete information game, the proposer offers a price and the responder chooses whether to trade. As before, agents who trade permanently leave the market, agents who fail to trade return to the searching population and the economy remains in steady state.

Under the assumption of the game under question, even if the proposer were allowed to choose any mechanism, she would choose the same take-it-or-leave-it offer as in the case where her type is publicly known (For a detailed argument see Atakan (2007) which adopts the development in Yilankaya (1999) to this framework). Consequently, the analysis is focused, without loss of generality, on a game-form where the proposer makes a take-it-or-leave-it offer and truthfully announces his/her type and the responder accepts or rejects the offer. A strategy σ_i for agent i specifies a transfer offer t and the truthful announcement of her type, if i is designated as the proposer, and a probability of accepting the price offer, if designated as the responder. The proposer can condition her offer on the measure of agents in the economy l and the observable characteristics of her partner for the period. As before, the match probability m_{bs} denotes the probability that b and s consummate their match and leave the market, given that they are paired in a period and b is chosen as the proposer. Also, as before, a search equilibrium is comprised of a mutually compatible steady state measure l and a strategy profile σ.

Private Information. A seller's type s specifies two variables: the good that the seller has for sale, denoted x_s, and the reservation value (or cost) for the seller, denoted r_s. For example, if the seller is an individual looking for a job, then x_s denotes the worker's specialization (e.g. engineer or manager) and r_s his or her disutility from labor. A buyer's type b also specifies two variables, the buyer's segment, denoted x_b, and the buyers reservation value, denoted r_b. For example, if the buyer is a firm attempting to hire a worker, then x_b denotes the type of vacancy that the firm is trying to fill, and r_b denotes the cost of making this job available or the production the firm can achieve if it were to staff the vacancy internally without hiring a new employee. If a buyer and seller match, then they create value $f_{bs} = h\left(x_s, x_b\right) - r_b - r_s$. Continuing with the example, when worker b is hired by firm s the total value created is given by the production achieved by hiring a worker with skill set x_s for job x_b, $h\left(x_s, x_b\right)$, net of the disutility from labor r_s and the cost for the firm r_b. In what follows, I write h_{bs} instead of $h(x_s, x_b)$ with the understanding that $h_{bs'} = h_{bs}$ for two sellers s and s' who own the same good, and likewise for buyers that belong to the same segment.

The analysis here assumes *independent private values*. That is, when a buyer and seller meet the buyer observes the good that a seller has for sale, i.e., x_s, while x_b, r_b and r_s may remain as private information. Also, *assume that agents know the steady state distribution and that any agent's prior belief about his/her trading partner's type coincides with the steady state distribution of types in the economy.*

The vector of values in a search equilibrium under private information satisfy the recursive equations

(16)

$$v_s = \left\{ \beta \sum_b p_b m_{bs}(t_{b,x_s} - r_s - v_s) + (1-\beta) \sum_b p_b m_{sb}(t_s - r_s - v_s) + v_s - c_S \right\}^+$$

(17)

$$v_b = \left\{ \beta \sum_s p_s m_{bs}(h_{bs} - t_{b,x_s} - r_b - v_b) + (1-\beta) \sum_s p_s m_{sb}(h_{bs} - t_s - r_b - v_b) + v_b - c_B \right\}^+ ,$$

where t_{b,x_s} represents the transfer offer made by buyer b to a seller endowed with good x_s and t_s the transfer offer made by a seller.

Before proceeding further, Theorem 2 is stated. This result shows that if search is costless, then the set of search equilibria under complete information and the set of equilibria under incomplete information coincide. That is, incomplete information of the independent private values variety is costly only in the presence of search frictions. In this model, with positive search costs, asymmetric information can cause inefficiencies because agents with strictly positive surplus may fail to consummate a match, that is, bargaining between buyers and sellers may be ex-post inefficient. However, when search frictions are absent, an agent has an incentive to wait for her favorite counterpart. This implies that the agent's option value adjusts and leaves no room for agreement with anybody but her favorite trading partner. Thus each agents ends up with his/her favorite match and the bargaining in the market is *ex-post efficient,* given the distribution of agents and the endogenously generated outside options.

Theorem 2. *Assume that $c_S = c_B = 0$. Also assume if seller s and buyer b meet, then x_s is observed by the buyer, however x_b, r_b and r_s are private information. The set of search equilibrium values and measures (v and l) under complete information and the set of search equilibria values and measures (v and l) under incomplete information are the same.*

Although the previous theorem focused on frictionless search, the main result of this section (Theorem 3) considers general search costs $c_B > 0$ and $c_S > 0$. In particular, Theorem 3 demonstrates, for any positive search cost, that the set of search equilibrium under complete information and the set of search equilibrium under complete information coincide, under the following assumption on the information structure.

Assumption 1 (Additively Separable Private Information (ASP))**.** *If seller s and buyer b meet, then x_s is observed by the buyer and x_b is observed by the seller however r_b and r_s are private information.*

In the case of additively separable private information, a buyer knows which good he/she is purchasing, but not the seller's cost r_s, while a seller observes the segment of the buyer, but not his/her reservation values r_b. In terms of the labor market example, the firm observes the skill set of the worker, but not his/her disutility from labor, and the worker observes the type of vacancy, but not the firm's cost. Under *ASP*, the seller can also condition the transfer offer t_s on the segment x_b of the buyer, i.e., t_{s,x_b}.

Before stating and proving Theorem 3, two preliminary results (Lemma 3 and 4) are needed. Lemma 3 shows that the difference between the option values of two buyers (or sellers), who belong to the same segment (who own the same good), is equal to the difference between their private reservation utility r_b (their private costs r_s). Lemma 4 employs the finding of Lemma 3 to prove a version of Proposition 1 under *ASP*, i.e., that the vector (l, m, v) comprises a search equilibrium under *ASP* if and only if it satisfies Conditions 1 through 4. Finally, Theorem 3 uses Lemma 4 and Proposition 1 to establish the equivalence between search equilibria under *ASP* and search equilibria under complete information.

Lemma 3. *Suppose that l, m and v satisfy Conditions 1 through 4, or alternatively that l is the steady state measure, m and v are the match probabilities and values associated with a search equilibrium under private information. If sellers s and s' own the same good ($x_s = x_{s'}$), then $v_s - v_{s'} = r_{s'} - r_s$ and thus $f_{bs} - v_b - v_s = f_{bs'} - v_b - v_{s'}$. Likewise, if buyers b and b' belong to the same segment($x_b = x_{b'}$), then $v_b - v_{b'} = r_{b'} - r_b$ and thus $f_{bs} - v_b - v_s = f_{b's} - v_s - v_{b'}$.*

Proof. Assume l, m and v satisfy Conditions 1 through 4. The Efficient Bargaining and Constant Surplus Conditions imply that the value to seller s' under the match probabilities $m_{bs'}$ must be at least as large under m_{bs}. This is because the match probability $m_{bs'} = 1$, if the surplus with buyer b is strictly positive and $m_{bs'} = 0$, if the surplus is strictly negative. Consequently,

$$c_S \geq (1-\beta) \sum\nolimits_b p_b m_{bs}(f_{bs'} - v_b - v_{s'}) \tag{18}$$

$$v_{s'} \geq \frac{-c_S + (1-\beta) \sum_b p_b m_{bs}(f_{bs'} - v_b)}{(1-\beta) \sum_b p_b m_{bs}} \tag{19}$$

Also,

$$v_s = \frac{-c_S + (1-\beta) \sum_b p_b m_{bs}(f_{bs} - v_b)}{(1-\beta) \sum_b p_b m_{bs}}. \tag{20}$$

Thus for sellers s' and s who own the same good

$$v_s - v_{s'} \leq \frac{\sum_b p_b m_{bs}(f_{bs} - f_{bs'})}{\sum_b p_b m_{bs}} = r_{s'} - r_s. \tag{21}$$

Reversing the role played by s and s' gives

$$v_s - v_{s'} \geq \frac{\sum_b p_b m_{bs'}(f_{bs} - f_{bs'})}{\sum_b p_b m_{bs'}} = r_{s'} - r_s, \tag{22}$$

and consequently $v_s - v_{s'} = r_{s'} - r_s$. Also, if b and b' belong to the same segment, then a similar argument as above shows that $v_b - v_{b'} = r_{b'} - r_b$.

Suppose that v and m represent values and match probabilities in a search equilibrium under private information. Rearranging the recursive equation for seller values, i.e., equation (16), gives

$$v_s = \frac{c_S + \beta \sum_b p_b m_{bs}(t_{b,x_s} - r_s) + (1-\beta)\sum_b p_b m_{sb}(t_{s,x_b} - r_s)}{\beta \sum_b p_b m_{bs} + (1-\beta)\sum_b p_b m_{sb}} \tag{23}$$

The value of seller s' must be at least as large as the value that this seller would get if she was to adopt strategy σ_s optimal for seller s who is endowed with the same good. Consequently,

$$v_{s'} \geq \frac{c_S + \beta \sum_b p_b m_{bs}(t_{b,x_s} - r_{s'}) + (1-\beta)\sum_b p_b m_{sb}(t_{s,x_b} - r_{s'})}{\beta \sum_b p_b m_{bs} + (1-\beta)\sum_b p_b m_{sb}}. \tag{24}$$

Thus, $v_s - v_{s'} \leq r_{s'} - r_s$. Exchanging the roles of seller s and s' implies that $v_s - v_{s'} \geq r_{s'} - r_s$ and consequently $v_s - v_{s'} = r_{s'} - r_s$. A symmetric argument establishes that $v_b - v_{b'} = r_{b'} - r_b$ for any two buyers who belong to the same segment. $\square$

Lemma 3 implies that if a buyer b accepts an offer t from seller s, that is, if $v_b \leq h(x_b, x_s) - r_b - t$, then all buyers b' from that segment should also be willing to accept the price offer, since $v_{b'} + r_{b'} - r_b = v_b \leq h(x_b, x_s) - r_b - t$. Likewise, if a seller accepts an offer t from buyer b, then all sellers who own the same good should also be willing to accept this price offer. Consequently, not knowing r_s or r_b does not constrain a proposer when making an offer. The proposer can extract all the surplus by simply conditioning the price offer on the good x_s or the segment x_b of the responder. The lemma below makes this line of reasoning exact and outlines its implications.

Lemma 4. *Assume ASP. If σ and l comprise a search equilibrium, then the equilibrium l, m and v satisfy Conditions 1 through 4. Conversely, if l, m and v satisfy Conditions 1 through 4, then there exists σ such that l and σ comprise a search equilibrium and equilibrium values are given by v.*

The following theorem shows that the set of search equilibria under *ASP* and search equilibria under perfect information are the same. Consequently, all results presented for perfect information apply, without alteration, to *ASP*.

Theorem 3. *The set of search equilibrium values and measures (v and l) under complete information and the set of search equilibria values and measures (v and l) under ASP are the same.*

Proof. If l and σ is a search equilibrium under complete information, then by Proposition 1 the equilibrium l, m and v satisfy Conditions 1 through 4. Thus, by Lemma 4, there exists $\hat{\sigma}$ such that l and $\hat{\sigma}$ are a search equilibrium under *ASP* and equilibrium values are given by v. Conversely, if l and σ comprise a search equilibrium under *ASP*, then by Lemma 4 the equilibrium l, m and v satisfy Conditions 1 - 4. Thus, by Proposition 1, there exists $\hat{\sigma}$ such that l and $\hat{\sigma}$ comprise a search equilibrium under complete information and equilibrium values are given by v. □

The equivalence established in Theorem 3 between search equilibria with perfect information and search equilibria under *ASP* is not true in the more general independent private values setting. Nevertheless, Theorem 2, stated and discussed at the beginning of this section, establishes the equivalence between search equilibria under complete and private information when there are no search friction, i.e., when $c_B = c_S = 0$. The following proof of Theorem 2 hinges on Proposition 1 and Lemma 5 (see the appendix). Lemma 5 is an exact analog of Lemma 4 that holds under the hypothesis of Theorem 2.

Proof of Theorem 2. If l and σ is a search equilibrium under complete information, then by Proposition 1 the equilibrium l, m and v satisfy Conditions 1 through 4. Thus, by Lemma 5, there exists $\hat{\sigma}$ such that l and $\hat{\sigma}$ are a search equilibrium under private information and equilibrium values are given by v. Conversely, if l and σ comprise a search equilibrium under private information, then by Lemma 5 the equilibrium l, m and v satisfy Conditions 1 through 4. Thus, by Proposition 1, there exists $\hat{\sigma}$ such that

l and $\hat{\sigma}$ comprise a search equilibrium under complete information and equilibrium values are given by v. □

3.3. **Convergence to a Competitive Equilibrium.** This subsection argues that search equilibria converge to competitive equilibria as search frictions disappear, that is, any limiting equilibrium is competitive (Theorem 4 and Corollary 4). Moreover, any search equilibria is also a competitive equilibria if there are no search frictions ($c_B = c_S = 0$), that is, equilibria at the limit are competitive (Corollary 5).

The competitive equilibrium benchmark considered here is a "flow" equilibrium as in Gale (1987) or Satterthwaite and Shneyerov (2007), generalized to accommodate heterogenous goods. In each period, flow supply is the measure of sellers of a particular good entering the market and flow demand is the measure of agents willing to purchase a particular good entering the market. In a flow equilibrium, the buyer and seller continuation values, which are the implicit prices, equate flow supply to flow demand for each good traded in the market. The competitive equilibrium allocations for economy $I = B \cup S$ is described by the following linear program which is the classical Assignment Problem where fractional assignments are permitted. As was shown in Proposition 2, the Assignment Problem and the planner's problem for $c_S = 0$ and $c_B = 0$, i.e., $\max_{l,m} W(I, 0, 0, l, m)$, are equivalent.

The Assignment Problem involves $\max_{q \geq 0} \sum_{B \times S} q_{bs} f_{bs}$ subject to $\sum_b q_{bs} \leq 1$ for all s and $\sum_s q_{bs} \leq 1$ for all b; while the dual of the Assignment problem involves $\min_{v \geq 0} \sum_B v_b + \sum_S v_s$ subject to $v_s + v_b \geq f_{bs}$ for all b and s. The vector q_{bs} that solves the program is a competitive allocation and denotes the measure of matches between b and s that are created in each period of time. Any vector v that solves the dual program is a competitive equilibrium utility vector and the competitive price of a traded good is $p_{x_s} = v_s + r_s$. The first constraint of the Assignment Problem states that the flow demand for seller of type s, i.e., $\sum_b q_{bs}$, must be less than the flow supply of that type, which is at most one. This constraint will bind, if the good's price is positive, or more precisely, if $v_s > 0$ and thus $p_{x_s} = v_s + r_s > r_s$. The second constraint states that the flow supply to buyers of type b, must be less than the flow demand by type b, which is at most one. Again, this constraint will bind if $v_b > 0$. Together the constraints ensure market clearing.

Theorem 4 and Corollary 4, presented below, consider sequences of search equilibria $(l(c^n), m(c^n), v(c^n))$ as search costs disappear (as $(c_B^n, c_S^n) \to 0$). Assume,

$$1 \leq \frac{c_S^n/1-\beta}{c_B^n/\beta} \leq \hat{r}$$

for all n, or in words, that the bargaining power of a buyer vis-à-vis a seller does not become arbitrarily large as search frictions disappear. This assumption ensures that any sequence $(l(c^n), m(c^n), v(c^n))$ is contained in a compact set and has a convergent subsequence. This is verified in the proof of Theorem 4. So in the statements of the results, attention is restricted, without loss of generality, to convergent subsequences.

Theorem 4 states that any sequence of search equilibria converge to a solution to the planner's problem with no search frictions for the economy H. Consequently, the limiting measure of matches is a competitive allocation and the limiting vector of values is a competitive utility vector for H. The set H is defined as the set of agents that are active in the market at the limit, i.e., the support of the limiting measure $l^n \to l$. In general H could be a proper subset of the actual economy I. For instance, since nobody entering the market is always a trivial equilibrium, the limit is a trivial competitive equilibrium for $H = \emptyset \subset I$ and consequently the limiting search equilibrium is inefficient. The two buyer, two seller example outlined in subsection 3.1 provides a more robust demonstration of a asymptotically inefficient sequence of search equilibria. Let $f_{12} = f_{21} = 1/5$ and $f_{11} = f_{22} = 1$, that is f is supermodular; buyer 1 likes seller 1's good and buyer 2 likes seller 2's good. For small c, and at the limit, efficiency requires that type 1 buyers trade only with type 1 sellers, and type 2's trade only with other type 2's. However, there is a sequence of stable search equilibria where only type 1 buyers and type 1 sellers enter the market. Consequently the limit of this sequence is inefficient with only the market for type 1 agents operating. Observe that for each c, the equilibrium is stable since forcing a small measure of type 2 buyers into the market would not entice type 2 sellers to flow suit and enter.

Nevertheless, Corollary 4, shows that there always exists a sequence of search equilibria, in particular a sequence of the "best" search equilibria as identified by Corollary 3, that converge to the competitive equilibrium of the full economy I and are consequently asymptotically efficient. Therefore, Theorem 4 (and Corollary 4) shows that inefficiencies caused by excess/insufficient entry or by incomplete information disappear along with search friction. However, inefficiencies caused by coordination failures may remain even at the frictionless limit.

Theorem 4. *If* $(l(c^n), m(c^n), v(c^n)) \to (\hat{l}, \hat{m}, \hat{v})$*, then* $(\hat{l}, \hat{m}) \in \arg\max_{l,m} W(H, 0, 0, l, m)$ *where* $H = supp(\hat{l})$*; and* $v_s + r_s$ *is a competitive equilibrium price vector for economy* H*.*

Proof. The sequence $(l(c^n), m(c^n), v(c^n))$ is contained in a compact set since $0 \le L_B^n \le L_s^n \le \hat{r}|B||S|$, $0 \le m_{bs} \le 1$ and $0 \le v(c^n) \le \bar{f}$. Thus a convergent subsequence exists. For a proof of $L_S^n \le \hat{r}|B||S|$ see the appendix.

It is straight forward to show that the vector $(\hat{l}, \hat{m}, \hat{v})$ satisfies Conditions 1 through 4 and consequently comprises a search equilibrium for I when $c_B = c_S = 0$. The following proves that any vector $(\hat{l}, \hat{m}, \hat{v})$ satisfying Conditions 1 through 4 solves the Assignment Problem for H and consequently is an element of $\arg\max_{l,m} W(H, 0, 0, l, m)$.

I show that the measure of per-period matches $\hat{q}_{bs} = \hat{m}_{bs}\hat{l}_b\hat{p}_s$ solves the Assignment Problem and $\hat{v}$ solves the dual of the Assignment Problem, consequently, that $\hat{q}$ is a competitive allocation and $\hat{v}_s - r_s$ is a competitive price vector for the economy H. Note that $\hat{q}$ is feasible for the Assignment Problem by Condition 4. Also, $\hat{v}$ is feasible for the dual of the Assignment Problem since, $\hat{s}_{bs}^+ \le 0$ and so $\hat{v}_b + \hat{v}_s \ge f_{bs}$ for all b and s. Consequently,

$$W \ge \sum_{B_H \times S_H} \hat{q}_{bs} f_{bs} = \sum_{B_H} \hat{v}_b + \sum_{S_H} \hat{v}_s \ge W \tag{25}$$

proving that $W = \sum_{B_H \times S_H} \hat{q}_{bs} f_{bs}$. □

Corollary 4. *There exists a sequence of search equilibria* $(l^n, m^n, v^n) \to (l, m, v)$ *such that*

$$(l, m) \in \arg\max W(I, 0, 0, l, m),$$

and consequently $v_s + r_s$ *is a competitive equilibrium price vector for the economy* I*.*

Proof. By Corollary 3 there exists (l^n, m^n, v^n) such that

$$(l_b^n/r^n, l_s^n, m^n) \in \arg\max W\left(c_S^n/2\,(1-\beta)\,, c_S^n/2\,(1-\beta)\,, l, m\right). \tag{26}$$

for all n. Let (l, m, v) denote the (possibly subsequential) limit. By Berge's Maximum Theorem $\arg\max W\left(I, c_S^n/2\,(1-\beta)\,, c_S^n/2\,(1-\beta)\,, l, m\right)$ is a compact valued, upper-semi-continuous correspondence. Consequently, the limit $(l_b/r, l_s, m) \in \arg\max W(I, 0, 0, l, m)$.

However, since there are no search costs also $(l_b, l_s, m) \in \arg\max W(I, 0, 0, l, m)$. By Proposition 2, $\max_{l,m} W(I, 0, 0, l, m)$ is equivalent to the Assignment Problem and so the per-period measure of matches $q_{bs} = p_s l_b m_{bs}$ is a competitive allocation and $v_s + r_s$ is competitive price vector for the economy I. □

Theorem 4 (and Corollary 4) showed that search equilibria are asymptotically competitive. The following corollary shows that the equilibria at the limit, i.e., when $c_B = c_S = 0$ are also competitive. Since by Theorem 2, if $c_B = c_S = 0$, the set of search equilibria under complete and incomplete information coincide, Corollary 5 covers both information structures.

Corollary 5. *Assume $c_B = c_S = 0$. If $(\hat{l}, \hat{m}, \hat{v})$ is a search equilibrium, then $(\hat{l}, \hat{m})$ is an element of* $\arg\max_{l,m} W(H, 0, 0, l, m)$ *where $H = supp(\hat{l})$; and $v_s + r_s$ is a competitive equilibrium price vector for economy H. Conversely, if $(\hat{l}, \hat{m})$ is an element of* $\arg\max_{l,m} W(I, 0, 0, l, m)$*, then $(\hat{l}, \hat{m}, \hat{v})$ is a search equilibrium.*

Proof. Theorem 4 implies that any $(\hat{l}, \hat{m}, \hat{v})$ that satisfies Condition 1 through 4 is an element of $\arg\max_{l,m} W(H, 0, 0, l, m)$. By Proposition 1 any search equilibrium satisfies Condition 1 through 4 and consequently is an element of $\arg\max_{l,m} W(H, 0, 0, l, m)$. Conversely, if $(\hat{l}, \hat{m})$ is and element of $\arg\max_{l,m} W(I, 0, 0, l, m)$, then $(\hat{l}, \hat{m}, \hat{v})$ is a search equilibrium by Theorem 1. □

3.4. **General Matching Functions.** This paper so far focused on a particular specification of the random matching process. The number of matches formed in each period is assumed to equal $\min\{L_B, L_S\}$. Previous literature, however, has considered a variety of matching structures. This subsection shows that the results, that depend on the particular specification of the random matching process (Theorem 1), are robust to different specifications of a matching function. Let $N(L_B, L_S)$ denote the number of matches between buyers and sellers formed in a period. The probability that a buyer meets seller s is

$$\frac{N(L_B, L_S)}{L_B} \frac{l_s}{L_S},$$

i.e., the probability that the buyer finds a partner $N(L_B, L_S)/L_B$, times the probability that this partner is actually seller s, l_s/L_S. Consequently, there are

$$\frac{N(L_B, L_S)\, l_s l_b}{L_B L_S}$$

matches formed between buyer b and seller s in a period.

Assumption 2 (MF)**.** *The match function $N : R_+^2 \to R_+$ is continuously differentiable, strictly concave, homogeneous of degree 1, and symmetric, i.e, $N(X, Y) = N(Y, X)$.*

The following theorem shows that if the buyers and sellers are symmetric in terms of both search costs and bargaining power, then an efficient equilibrium exists.

Proposition 3. *Suppose that the matching function satisfies Assumption (MF), $c_B = c_S$ and $\beta = 1/2$. A vector (l, m, v) comprises a search equilibrium if and only if it satisfies the first order conditions for the planner's problem. Consequently, if $(l, m) \in \arg\max_{l,m} W(c_B, c_S, l, m)$, then (l, m, v) comprises an efficient search equilibrium, where v_b and v_s are the Kuhn-Tucker multipliers associated with steady state constraints of the planner's problem.*

4. Discussion and Conclusion

This paper considered a frictional market where buyers and sellers, with unit demand and supply, search for trading opportunities. The analysis focused on explicit search frictions, allowed for two-sided incomplete information. In this context, a non-trivial, full trade search equilibrium was shown to exist, equilibria characterized as the values that satisfy the first order conditions for a non-linear planner's (optimization) problem, and necessary and sufficient conditions provided for the existence of efficient search equilibria under complete information. Under an additive separability condition, these results were shown to generalize to the two-sided incomplete information setting. Also, search equilibria were shown to converge to a competitive equilibria as frictions become small. However, inefficiencies could remain at the limit: the limiting competitive equilibrium can be an equilibrium for only a proper subset of the full economy under consideration. Nevertheless a sequence of search equilibria that converges to the competitive equilibrium of the full economy was also shown to exist.

Appendix A. Omitted Proofs

A.1. **Proof of Proposition 1.** Define a strategy profile σ such the proposer i offers utility v_j to the responder, if $f_{ij} - v_i - v_j \geq 0$ and demands utility v_i, if $f_{ij} - v_i - v_j < 0$

and the responder accepts with probability m_{ij}. Also, strategy profile σ prescribes that all agents of type i enter if $v_i > 0$, a measure $l_i \sum_j m_{ij} p_j$ enters if agents of that type are indifferent to entering and no agents enter if they strictly prefer not to enter. This strategy is sub-game perfect, solves the maximization problem for each agent and the market remains in steady state. Also, due to the constant surplus conditions, v is indeed the value for each type implied by σ and l.

A.2. **Proof of Lemma 1.** I also prove that if $c_B \geq 0$ and $c_S \geq 0$, then the maximized value of the problem with the additional constraint $L_S = L_B$, equals the maximized value for the planner's problem.

Assume $c_B > 0$ and $c_S > 0$. To show $L_S = L_B$ suppose to the contrary that $L_S > L_B$. By definition, the number of matches that can be formed between buyer b and seller s is limited by the steady state number of matches formed between the two types, i.e., $l_b l_s / \max\{L_S, L_B\} = l_b l_s / L_S$. However, scaling down the measure of each type of seller in the market by L_B/L_S will leave the number of matches between any two types b and s constant while decreasing the search costs by $(L_S - L_B)\, c_S$ and thus showing that $L_S = L_B$ at an optimum. If on the other hand $c_S = 0$, then it is possible that a steady state measure l with $L_S > L_B$ solves the planner's problem. However then, the measure $\hat{l}$ obtained by scaling down the sellers by L_B/L_S is also a maximizer since this scaling neither affects the search costs nor the measure of per period matches that are formed.

A.3. **Proof of Proposition 2.** Suppose that $\max\{c_B, c_S\} > 0$. The maximum number of matches formed in a period is bounded above by the maximum number of agents entering the economy, i.e., $\max\{|B|, |S|\}$. So, the maximum production in a period is bounded by $(\max_{b,s} f_{bs})\,(\max\{|B|, |S|\})$. Consequently, any solution of the planner's problem must satisfy

$$L_B c_B + L_S c_S \leq (\max\nolimits_{b,s} f_{bs})\,(\max\{|B|, |S|\})\,. \tag{27}$$

Imposing the inequality $L_B c_B + L_S c_S \leq (\max_{b,s} f_{bs})\,(\max\{|B|, |S|\}) + \epsilon$, with $\epsilon > 0$, in addition to $L_B = L_S$ to the planner's problem ensures that the constraint set for the planner's problem is compact and the Weierstrass' Theorem implies that a maximizer exists. It is straight forward to see that any optimizer for this problem with the additional constraints, also solves the planner's problem. The first additional constraint $L_B c_B + L_S c_S \leq (\max_{b,s} f_{bs})\,(\max\{|B|, |S|\}) + \epsilon$ is never binding because, if it were to bind, this would imply that the maximum value is strictly negative. However, choosing $L_S = L_B = 0$ ensures a non-negative value. Also, Lemma 1 showed that imposing the constraint $L_B = L_S$ does not affect the value of the maximization problem.

Suppose that $c_B = c_S = 0$. Let q solve

$$P = \max_q \sum\nolimits_{B\times S} q_{bs} f_{bs} \tag{28}$$

subject to

$$\sum_b q_{bs} \le 1, \tag{29}$$

$$\sum_s q_{bs} \le 1, \tag{30}$$

$$0 \le q_{bs} \le 1. \tag{31}$$

This is a well defined linear program and consequently has a solution. In particular, this is exactly the Assignment Problem. Let q solve the above linear program and pick any l such that $l_b p_s \ge q_{bs}$ for all b and s and choose m_{bs} to ensure that $m_{bs} l_b p_s = q_{bs}$. The chosen m and l is feasible for the maximization problem $W(0,0,l,m)$ hence $W(0,0,l,m) \ge P$. However, any $q_{bs} = m_{bs} l_b p_s$ for $(l,m) \in \arg\max_{l,m} W(0,0,l,m)$ is feasible for the above problem so $W(0,0,l,m) \le P$. This implies that if q is a maximizer of the above problem and $m_{bs} l_b p_s = q_{bs}$ for all b and s then $(l,m) \in \arg\max_{l,m} W(0,0,l,m)$ showing that $\arg\max_{l,m} W(0,0,l,m) \ne \emptyset$.

A.4. **Proof of Theorem 1.** The hypothetical planner's problem can be reformulated, under the assumption that $L_B = L_S = L$ as follows:

$$\max_{L,p,m} \sum\nolimits_{B\times S} L p_b p_s m_{bs} f_{bs} - L \frac{c_S}{1-\beta} \tag{32}$$

subject to

$$\sum\nolimits_s Lp_b p_s m_{bs} \le 1 \text{ for all } i \qquad (v_b) \tag{33}$$

$$\sum\nolimits_b Lp_b p_s m_{bs} \le 1 \text{ for all } j \qquad (v_s) \tag{34}$$

$$\sum\nolimits_b p_b = 1 \qquad (\mu_B) \tag{35}$$

$$\sum\nolimits_s p_s = 1 \qquad (\mu_S) \tag{36}$$

$$m_{bs} \le 1 \text{ for all } b \text{ and } s \qquad (\omega_{bs}) \tag{37}$$

$$m_{bs} \ge 0 \text{ for all } b \text{ and } s \qquad (\chi_{bs}) \tag{38}$$

$$p_b \ge 0 \text{ for all } b \qquad (\gamma_b) \tag{39}$$

$$p_s \ge 0 \text{ for all } s \qquad (\gamma_s) \tag{40}$$

The Lagrangian for the problem,

$$\begin{aligned} \mathcal{L} = -L\frac{c_S}{1-\beta} + \sum\nolimits_{B\times S} Lp_b p_s m_{bs} f_{bs} + \sum\nolimits_b v_b \left(1 - \sum\nolimits_s Lp_b p_s m_{bs}\right) \\ + \sum\nolimits_s v_s \left(1 - \sum\nolimits_b Lp_b p_s m_{bs}\right) + \mu_B \left(1 - \sum\nolimits_b p_b\right) + \mu_S \left(1 - \sum\nolimits_s p_s\right) \\ + \sum\nolimits_{B\times S} \omega_{bs} (1 - m_{bs}) + \sum\nolimits_{B\times S} \chi_{bs} m_{bs} + \sum\nolimits_b \gamma_b p_b + \sum\nolimits_s \gamma_s p_s \end{aligned} \tag{41}$$

where v_b, v_s, ω_{bs}, χ_{bs}, γ_b and γ_s are non-negative. The existence of a solution (L, m, p) to the above maximization problem follows from Proposition 2. First I show if (L, m, p, v) satisfy the first order conditions for the maximization problem, then $l_b = rp_b L$, $l_s = p_s L$, m and multipliers v_b and v_s satisfy Conditions 1 through 4 and therefore comprise a search equilibrium.

The vector v is non-negative since its elements are multipliers associated with inequality constraints and thus the Individual Rationality condition is satisfied.

Taking the first order condition with respect to m_{bs} and rearranging gives

$$Lp_b p_s f_{bs} - Lp_b p_s v_b - Lp_b p_s v_s - \omega_{bs} + \chi_{bs} = 0 \tag{42}$$

$$Lp_b p_s (f_{bs} - v_b - v_s) = \omega_{bs} - \chi_{bs} \tag{FOC m}$$

Taking the first order condition with respect to p_b implies

$$\sum\nolimits_s Lp_s m_{bs} f_{bs} - v_b \sum\nolimits_s Lp_s m_{bs} - \sum\nolimits_s v_s Lp_s m_{bs} - \mu_B + \gamma_b = 0 \tag{43}$$

$$\sum\nolimits_s p_s m_{bs} \left(f_{bs} - v_b - v_s\right) = \frac{\mu_B - \gamma_b}{L}, \tag{FOC p_b}$$

and likewise for seller s:

$$\sum\nolimits_b p_b m_{bs} \left(f_{bs} - v_b - v_s\right) = \frac{\mu_S - \gamma_s}{L}. \tag{FOC p_s}$$

The first order condition with respect to L is

$$\sum\nolimits_{B\times S} p_b p_s m_{bs} \left(f_{bs} - v_b - v_s\right) = \frac{c_S}{1-\beta} \tag{FOC L}$$

The complementary slackness conditions for the problem are

$$v_b \left(1 - \sum\nolimits_s Lp_b p_s m_{bs}\right) = 0, \tag{44}$$

$$v_s \left(1 - \sum\nolimits_b Lp_s p_b m_{bs}\right) = 0, \tag{45}$$

$$\omega_{bs} \left(1 - m_{bs}\right) = 0, \tag{46}$$

$$\chi_{bs} m_{bs} = 0, \tag{47}$$

$$\gamma_b p_b = 0, \tag{48}$$

$$\gamma_s p_s = 0. \tag{49}$$

I show, for agents with $p_b > 0$, equation (FOC m) and complementary slackness conditions equations (46) and (47) together imply the Efficient Bargaining Condition. Assume $f_{bs} - v_b - v_s < 0$. This implies that $Lp_b p_s \left(f_{bs} - v_b - v_s\right) - \omega_{bs} < 0$, consequently $\chi_{bs} > 0$ and thus $m_{bs} = 0$. Assume that $f_{bs} - v_b - v_s > 0$. This implies that $\omega_{bs} > 0$ and consequently $m_{bs} = 1$.

I show that the first order condition with respect to p and L in conjunction with complementary slackness conditions (48) and (49) imply the Constant Surplus Condition. If $p_b > 0$, equation (48) implies that $\gamma_b = 0$. Consequently substituting μ_B/L for $\sum_s p_s m_{bs} \left(f_{bs} - v_b - v_s\right)$ for agents with $p_b > 0$ into equation (FOC L) delivers

$$\sum\nolimits_b p_b \sum\nolimits_s p_s m_{bs} \left(f_{bs} - v_b - v_s\right) = \frac{c_S}{1-\beta} \tag{50}$$

$$\sum\nolimits_b p_b \frac{\mu_B}{L} = \frac{c_S}{1-\beta} \tag{51}$$

and consequently $\mu_B/L = \frac{c_S}{1-\beta}$. Going through the same steps for a seller shows that $\mu_S/L = \frac{c_S}{1-\beta}$. Substituting for μ_B and μ_S in (FOC p_b) and (FOC p_s) implies that

$$\sum\nolimits_s p_s m_{bs}\left(f_{bs} - v_b - v_s\right) \leq \frac{c_S}{1-\beta} \text{ for all buyers } b, \tag{52}$$

$$\sum\nolimits_b p_b m_{bs}\left(f_{bs} - v_b - v_s\right) \leq \frac{c_S}{1-\beta} \text{ for all seller } s. \tag{53}$$

For buyers and sellers with $p_b > 0$ and $p_s > 0$, the previous inequalities holds with equality. Dividing both sides or equation (80) by r shows that $\sum_s p_s/r m_{bs}\left(f_{bs} - v_b - v_s\right) \leq c_B/\beta$ and thus the Constant Surplus Condition is satisfied.

By construction $Lp_b \sum_s m_{bs}p_s \leq 1$. Also, equation (44) implies that if $v_b > 0$, then $1 - Lp_b \sum_s m_{bs}p_s = 0$ and likewise for a seller. Consequently, equations (44) and (45) imply the Steady State Condition.

Now I show if (l, m, v) is a search equilibrium, then it satisfies the first order and complementary slackness conditions for the planner's problem. Let v be the multipliers for the steady state constraints and let $\mu_S/L = c_S/(1-\beta)$, $\mu_B/L = c_B/\beta$, $\chi_{bs} = \max\{-s_{bs}, 0\}$, $\omega_{bs} = \max\{s_{bs}, 0\}$, $\gamma_b = c_B - \beta \sum p_s m_{sb} s_{bs} \geq 0$ and $\gamma_s = c_S - (1-\beta)\sum p_b m_{sb} s_{bs} \geq 0$. Given these definitions, it is straight forward to verify that the first order and complementary slackness conditions are all satisfied.

Finally, I show the existence of Kuhn-Tucker multipliers, that satisfy the first order conditions, at any maximizer of the planner's problem. In order to show that Kuhn-Tucker multipliers exists, I show that the Mangasarian-Fromovitz constraint qualification is satisfied. The Mangasarian-Fromovitz constraint qualification can be stated as follows: Assume x^* solves $\max_x f(x)$ subject to inequality constraints, $g_i(x) \geq 0$, $i = 1, ..., k$, and equality constraints, $h_j(x) = 0$, $j = 1, ..., m$. If the gradients of the equality constraints, i.e., $\nabla h_j(x^*)$, $j = 1, ..., m$, are linearly independent and there exists a vector d such that $\nabla g_i(x^*)^T d > 0$ for all binding inequality constraints and $\nabla h_j(x^*)^T d = 0$ for all equality constraints, then there exists a vector $\mu \neq 0$ such that $\nabla f(x^*) + \sum_i \mu_i \nabla g_i(x^*) + \sum_j \mu_j \nabla h_j(x^*) = 0$ (see Bertsekas (2003) for further detail).

In the current setting the gradient of a constraint g_i is a column vector of the following form

(54)

$$\nabla g_i\left(L,p,m\right)=\left(\frac{\partial g_i}{\partial L},\frac{\partial g_i}{\partial p_{b1}},...,\frac{\partial g_i}{\partial p_b},...,\frac{\partial g_i}{\partial p_{|B|}},\frac{\partial g_i}{\partial p_{s1}},...,\frac{\partial g_i}{\partial p_s},...,\frac{\partial g_i}{\partial p_{|S|}},\frac{\partial g_i}{\partial m_{11}},...,\frac{\partial g_i}{\partial m_{bs}},...,\frac{\partial g_i}{\partial m_{|B||S|}}\right)$$

The planner's problem has two equality constraints; $1-\sum_b p_b=0$ and $1-\sum_s p_s=0$. The gradients of these constraints are of the following form

$$\nabla\left(1-\sum_b p_b\right)=(0,-1,...,-1,...,-1,0,...,0,...,0,0,...,0,...,0) \tag{55}$$

$$\nabla\left(1-\sum_s p_s\right)=(0,0,...,0,...,0,-1,...,-1,...,-1,0,...,0,...,0) \tag{56}$$

and consequently clearly linearly independent. Assume without loss of generality that, $p_{b_1}>0$ and $p_{s_1}>0$, i.e., the non-negativity constraint is not binding for the first buyer and the first seller. I will now construct the vector d. Pick the entry in the vector d that corresponds to p_b equal to $\varepsilon>0$ if $p_b=0$ and equal to 0, if $p_b>0$ for all $b\neq 1$. Pick the entry for p_{b_1} to ensure that $\nabla\left(1-\sum_b p_b\right)^T d=0$. Proceed similarly for the p_s entries and consequently $\nabla\left(1-\sum_s p_s\right)^T d=0$ Pick the entry in d that corresponds to m_{bs} equal to ε if $m_{bs}=0$, equal to $-\varepsilon$, if $m_{bs}=1$ and equal to zero otherwise. Observe that all entries in the vector d, except d_1 the entry that corresponds to L, are now determined. The inner product between d and the gradient of binding non-negativity constraints are equal to ε and consequently strictly positive. Also, for a biding constraint of the form $1-m_{bs}\geq 0$, the inner product is also equal to ε and consequently positive. The only constraints left to check are of the form $1-\sum_b\Lambda p_b p_s m_{bs}\geq 0$ for seller s or $1-\sum_s\Lambda p_b p_s m_{bs}\geq 0$ for buyer b. For example, if buyer $1's$ constraint is binding, then the gradient is as follows

(57)

$$-\left(\sum_s m_{1s}p_{b_1}p_s,\sum_s Lm_{1s}p_s,...,0,...,0,Lm_{11}p_{b_1},...,Lm_{1s}p_{b1},...,Lm_{1|S|}p_{b_1},Lp_{b_1}p_{s_1},...,0,...,0\right)=$$
$$-\left(\frac{1}{L},\sum_s Lm_{1s}p_s,...,0,...,0,Lm_{11}p_{b_1},...,Lm_{1s}p_{b1},...,Lm_{1|S|}p_{b_1},Lp_{b_1}p_{s_1},...,0,...,0\right)$$

and $\nabla\left(1-\sum_s Lp_{b1}p_s m_{1s}\right)^T d=-\frac{d_1}{L}-\varepsilon A$. Consequently, we can pick d_1 sufficiently negative in order to make $\nabla\left(1-\sum Lp_b p_s m_{bs}\right)^T d>0$ for all b and s.

A.5. **Proof of Corollary 2.** Fix l_s to equal the search equilibrium measure of sellers and consider the linear program

$$P = \max_{p_b, q} \sum\nolimits_{B \times S} q_{bs} f_{bs} - L_S \left(\sum\nolimits_b p_b \right) c_B \tag{58}$$

subject to

$$\sum\nolimits_s q_{bs} \leq 1 \text{ for all } b, \tag{59}$$

$$\sum\nolimits_b q_{bs} \leq 1 \text{ for all } s, \tag{60}$$

$$q_{bs} - p_b l_s \leq 0 \text{ for all } b \text{ and } s, \tag{61}$$

$$\sum\nolimits_b p_b \leq 1, \tag{62}$$

$$q_{bs} \geq 0, \tag{63}$$

$$p_b \geq 0. \tag{64}$$

Claim 1. *If q and p_b solve the above linear program, then any m_{bs} defined to satisfy $q_{bs} = m_{bs} p_b l_s$, and $l_b = p_b L_S$ is an element of $\arg\max_{l_b, m} W(c_B, c_S, l_b.l_s, m)$.*

Assume that q and p_b solve the above linear program, then m_{bs} defined to satisfy $q_{bs} = m_{bs} p_b l_s$ and $l_b = p_b L_S$ is feasible for the maximization problem $W(c_B, c_S, l_b.l_s, m)$, hence $\max_{l_b, m} W(c_B, c_S, l_b.l_s, m) \geq P$.

If $l_b, m_{bs} \in \arg\max_{l_b, m} W(c_B, c_S, l_b.l_s, m)$, then $l_b \leq l_s$ since if $l_b > l_s$ we could scale the measure of buyers down without effecting the value. This implies that $p_b = l_b / L_S$ and $q_{bs} = m_{bs} p_b l_s$ is feasible of the above linear program and so $P \geq \max_{l_b, m} W(c_B, c_S, l_b.l_s, m)$. Showing that $P = \max_{l_b, m} W(c_B, c_S, l_b.l_s, m)$ and proving the claim.

Claim 2. *The search equilibrium distribution of buyers p_b and measure of matches $q_{bs} = m_{bs} p_b l_s$ solves the linear program.*

The search equilibrium proportion of buyers p_b and measure of matches $q_{bs} = m_{bs} p_b l_s$ is feasible for the above program consequently $\sum_{B \times S} m_{bs} p_b l_s f_{bs} - L_S c_B \leq P$. The dual to the above linear program is as follows:

$$D = \min_{\nu, \mu, \omega} \sum\nolimits_b \nu_b + \sum\nolimits_s \nu_s + \mu \tag{65}$$

subject to

$$\nu_b + \nu_s + \omega_{bs} \geq f_{bs} \text{ for all } b \text{ and } s, \tag{66}$$

$$\mu - \sum\nolimits_s l_s \omega_{bs} \geq -c_B L_S \text{ for all } b, \tag{67}$$

$$\nu_i \geq 0 \text{ for all } i \in I, \tag{68}$$

$$\omega_{bs} \geq 0 \text{ for all } b \text{ and } s. \tag{69}$$

The search equilibrium values v_b, v_s, the surplus $s_{bs}^+ = \max\{f_{bs} - v_s - v_b, 0\}$ and $\mu = c_S L_S$ are feasible for the dual since $v_b + v_s + s_{bs}^+ \geq f_{bs}$ for all b and s and due to the constant surplus condition $\sum_s l_s s_{bs}^+ = L_S \sum_s p_s s_{bs}^+ \leq (c_B + c_S) L_S = \frac{c_B}{\beta} L_S$ for all b. This implies that $\sum_b v_b + \sum_s v_s + c_S L_S \geq D$. However, $\sum_b v_b + \sum_s v_s + c_S L_S = \sum_{B\times S} m_{bs} p_b l_s f_{bs} - c_B L_S \geq D = P$. This implies that$\sum_{B\times S} m_{bs} p_b l_s f_{bs} - c_B L_S = P$ showing that the search equilibrium distribution of buyers p_b and measure of matches $q_{bs} = m_{bs} p_b l_s$ solves the linear program.

Define $l_b = p_b \sum_s l_s$. Claim 1 and Claim 2 together imply that

$$(l_b, m) \in \arg\max\nolimits_{l_b,m} W(c_B, c_S, l_b.l_s, m). \tag{70}$$

The argument for $(l_s, m) \in \arg\max_{l_s,m} W(c_B, c_S, l_b.l_s, m)$ is symmetric.

A.6. **Proof of Lemma 4.** Suppose that l, m and v satisfy Conditions 1 through 4. I show that there exists a search equilibrium (σ, l) where values are given by v and the match probabilities by m.

Define a strategy profile σ as follows: Let the number of agents of type i entering the economy equal $l_i \sum_j p_j m_{ij}$. The proposer truthfully announces his/her type to the responder and makes a transfer offer. For a buyer b let the transfer offer made to a seller of good $\hat{x}$

$$t_{b,\hat{x}} = v_{s'} + r_{s'}, \qquad \text{if } f_{bs'} - v_b - v_{s'} \geq 0, \text{ and} \tag{71}$$

$$t_{b,\hat{x}} = h_{bs'} - r_b - v_b, \qquad \text{if } f_{bs'} - v_b - v_{s'} < 0, \tag{72}$$

where s' is any seller with $x_{s'} = \hat{x}$. By Lemma 3, $v_{s'} + r_{s'} = v_s + r_s$ and $f_{bs'} - v_b - v_{s'} = f_{bs} - v_b - v_s$ for all sellers s' and s who own the same good $\hat{x}$. For a seller s let the transfer offer made to any buyer who belongs to segment $\hat{x}$

$$t_{s,\hat{x}} = h_{b's} - r_{b'} - v_{b'}, \qquad \text{if } f_{b's} - v_s - v_{b'} \geq 0, \text{ and} \tag{73}$$

$$t_{s,\hat{x}} = v_s + r_s, \qquad \text{if } f_{b's} - v_s - v_{b'} < 0, \tag{74}$$

where b' is any buyer with $x_{b'} = \hat{x}$. Again by Lemma 3, $h_{b's} - r_{b'} - v_{b'} = h_{bs} - r_b - v_b$ and $f_{b's} - v_s - v_{b'} = f_{bs} - v_b - v_s$ for all buyers b' and b who belong to the same segment $\hat{x}$. Let buyer b accept an offer from a seller s with probability m_{sb} and the seller accept an offer from the buyer with probability m_{bs}.

The strategy profile σ defined above and l is compatible with a steady state since the match probabilities that I used to define this strategy profile, satisfy the Steady State Condition by assumption. Since a buyer matches with any seller with whom she has positive surplus and extracts all the rent when she is a proposer and gets her continuation value when she is responder, her decisions are optimal, and likewise for sellers. Thus no agent has an incentive to deviate from the prescribed strategies. Also, since the revelation of type does not affect the matching decision when the surplus is strictly positive, the proposers do not have an incentive to misreport.

Now I show that if σ and l is a search equilibrium under ASP , then l, m and v satisfy Conditions 1 through 4. Observe that the strategy profile σ is available to agents under complete information. Since l, m and v are compatible with equilibrium, values satisfy the Individual Rationality Condition and match probabilities and the steady state measure satisfy the Steady State Condition.

By Lemma 3, a proposer can extract all surplus from a responder by conditioning only on the responder's segment, if the responder is a buyer, and conditioning only on the good, if the responder is a seller. If $s_{bs} = f_{bs} - v_s - v_b > 0$, then since the proposer can extract all the surplus, a meeting between such a pair must result in a certain match, i.e., $m_{bs} = 1$. Also, if $s_{bs} < 0$, then the offer made by the proposer can not satisfy the responder and thus $m_{bs} = 0$. Consequently, the Efficient Bargaining Condition holds.

By the reasoning in the previous paragraph, if $s_{bs} > 0$, then $t_{s,x_b} = h_{bs} - r_b - v_b$ and $t_{b,x_s} = r_s + v_s$. Substituting these transfers into the recursive equation for values gives

$$v_s = \max\left\{-c_S + (1-\beta)\sum_b p_b m_{sb}(h_{bs} - r_b - v_b - r_s - v_s) + v_s, 0\right\} \text{ and} \tag{75}$$

$$v_b = \max\left\{-c_B + \beta\sum_s p_s m_{bs}(h_{bs} - r_s - v_s - r_b - v_b) + v_b, 0\right\}, \tag{76}$$

and rearranging shows that the Constant Surplus Condition holds.

A.7. **Proof of Theorem 2.**

Lemma 5. *Assume that $c_S = c_B = 0$. Also, assume if seller s and buyer b meet, then x_s is observed by the buyer, however x_b, r_b and r_s are private information. If σ and l comprise a search equilibrium, then the equilibrium l, m and v satisfy Conditions 1 through 4. Conversely, if l, m and v satisfy Conditions 1 through 4, then there exists σ such that l and σ comprise a search equilibrium and equilibrium values are given by v.*

Assume l, m and v satisfy Conditions 1 through 4. I outline strategy $\hat{\sigma}$ such that $\hat{\sigma}$ and l comprise a search equilibrium under private information where values are given by v. The Constant Surplus Condition implies that

$$\sum_j p_j m_{ij}(f_{ij} - v_i - v_j) = 0 \tag{77}$$

for all i and the Efficient Bargaining Condition implies that if $m_{bs} > 0$, then $f_{bs} - v_b - v_s \geq 0$. Consequently, $f_{bs} - v_b - v_s \leq 0$ for all b and s, and $f_{bs} - v_b - v_s = h_{bs} - r_s - r_b - v_b - v_s = 0$ for b and s with $m_{bs} > 0$.

Define strategy $\hat{\sigma}$ such that a seller, when chosen as the proposer, proposes to sell their good for $t = v_s + r_s$ and truthfully announce his/her type s to the responder. The buyer responds by accepting the offer with probability m_{sb}. By Lemma 2, all sellers endowed with the same good propose the same transfer. Also, the specified transfer is the highest amount any buyer would be willing to pay for the particular good, given values v. Consequently, the seller has neither an incentive to misrepresent her type nor to deviate from the prescribed transfer offer. Also, a buyer cannot gain by deviating, since for a buyer with $m_{sb} > 0$, the proposed transfer leaves the buyer indifferent between accepting and rejecting.

When a buyer proposes, he/she offers to pay $t = h_{bs} - r_b - v_b$ and announces his/her type b truthfully. The seller responds by accepting with probability m_{bs}. Observe that all buyers who have a non-negative surplus with seller s, offer the same transfer and this is the lowest transfer that will be accepted by any seller who owns the same good as seller s. Moreover, the buyer cannot get more surplus than under the specified transfer offer, given values v. Consequently, the buyer has neither an incentive to misrepresent her type nor to deviate from the prescribed transfer offer. Also, a seller cannot gain by deviating, since for a seller with $m_{bs} > 0$, the proposed transfer leaves the seller indifferent between accepting and rejecting.

Since the matching probabilities and the steady state measure satisfy the Steady State condition, the market is in a steady state. Also, using the strategies above implies that the values are given by v since the transfers and acceptance decisions are identical to the full information case.

I show equilibrium l, m and v satisfy Conditions 1 through 4. The Individual Rationality and the Steady State Conditions are obviously satisfied. The Bellman equation in equilibrium are

$$0 = \beta \sum_b p_b m_{bs}(t_{b,x_s} - r_s - v_s) + (1-\beta) \sum_b p_b m_{sb}(t_s - r_s - v_s) \text{ and} \tag{78}$$

$$0 = \beta \sum_s p_s m_{bs}(h_{bs} - t_{b,x_s} - r_b - v_b) + (1-\beta) \sum_s p_s m_{sb}(h_{bs} - t_s - r_b - v_b) \tag{79}$$

Consequently, if $m_{bs} > 0$, then $t_{b,x_s} - r_s - v_s = 0$ and $h_{bs} - t_{b,x_s} - r_b - v_b = 0$, also if $m_{sb} > 0$, then $h_{bs} - t_s - r_b - v_b = 0$ and $t_s - r_s - v_s = 0$. Substituting gives

$$0 = (1-\beta) \sum_b p_b m_{sb}(h_{bs} - r_s - v_s - r_b - v_b) \text{ and} \tag{80}$$

$$0 = \beta \sum_s p_s m_{bs}(h_{bs} - r_s - v_s - r_b - v_b) \tag{81}$$

showing that the Constant Surplus Condition is satisfied.

To show that the Efficient Bargaining Condition holds, I show that $f_{bs} - v_s - v_b \leq 0$ for all b and s. Assume that there exists a pair b and s such that $f_{bs} - v_s - v_b > 0$. Then the proposer can demand $v_i + \epsilon$ while ensuring the acceptance by giving the responder $v_j + \epsilon$. The proposer will always do so in a perfect equilibrium consequently $m_{ij} = 1$. However, this contradicts the Constant Surplus Condition that was shown to hold in equilibrium since the pair b and s meet with strictly positive probability.

A.8. **Proof of Theorem 4.** For any c, search equilibrium p and L solve the optimization problem

$$\min_{p_b, p_s, L} 2cL \tag{82}$$

subject to

$$q_{bs} \le p_b p_s L \tag{83}$$

$$\sum p_b = 1 \tag{84}$$

$$\sum p_s = 1 \tag{85}$$

where q_{bs} is a fixed and equals the number of b and s matches that are formed in a search equilibrium, i.e., $q_{bs} = m_{bs} p_b p_s L$. Showing that search equilibrium p and L solve the problem is immediate since the surplus values s, p and L satisfy the first order conditions, the inequality constraints are quasiconcave and the equality constraints as well as the maximized function are linear. However, observe that $p_b = \frac{1}{|B|}$, $p_s = \frac{1}{|S|}$ and $L = |B| \times |S|$ is always a feasible option. Consequently, $2cL \le 2c\left(|B| \times |S|\right)$ and hence $L \le |B| \times |S|$.

A.9. **Proof of Proposition 3.** The Lagrangian for the problem is

(86)

$$\begin{aligned}\mathcal{L} = -\left(L_B + L_S\right) c + N\left(L_B, L_S\right) \sum\nolimits_{B\times S} p_b p_s m_{bs} f_{bs} + \sum\nolimits_b v_b \left(1 - N\left(L_B, L_S\right) \sum\nolimits_s p_b p_s m_{bs}\right) \\ + \sum\nolimits_s v_s \left(1 - N\left(L_B, L_S\right) \sum\nolimits_b p_b p_s m_{bs}\right) + \mu_B \left(1 - \sum\nolimits_b p_b\right) + \mu_S \left(1 - \sum\nolimits_s p_s\right) \\ + \sum\nolimits_{B\times S} \omega_{bs} \left(1 - m_{bs}\right) + \sum\nolimits_{B\times S} \chi_{bs} m_{bs} + \sum\nolimits_b \gamma_b p_b + \sum\nolimits_s \gamma_s p_s.\end{aligned}$$

The concavity of N implies that $L_B = L_S$. I show that any vector (l, m, v) that satisfies the first order conditions also satisfy Conditions 1 through 4. The first order conditions with respect to m is

$$N\left(L_B, L_S\right) p_b p_s \left(f_{bs} - v_b - v_s\right) = \omega_{bs} - \chi_{bs}. \tag{87}$$

This first order condition implies the Efficient Bargaining Condition. The first order conditions with respect to p_b is

$$N\left(L_B, L_S\right) \sum\nolimits_s p_s m_{bs} \left(f_{bs} - v_b - v_s\right) = \mu_B - \gamma_b \tag{88}$$

and likewise for p_s

$$N(L_B, L_S)\sum\nolimits_b p_b m_{bs}(f_{bs} - v_b - v_s) = \mu_S - \gamma_s \tag{89}$$

The first order conditions with respect to L_B and L_S are

$$N_B(L_B, L_S)\sum\nolimits_{B\times S} p_b p_s m_{bs}(f_{bs} - v_s - v_b) = c \tag{90}$$

$$N_S(L_B, L_S)\sum\nolimits_{B\times S} p_b p_s m_{bs}(f_{bs} - v_s - v_b) = c \tag{91}$$

Substituting gives $\mu_B = \frac{N(L_B,L_S)}{N_B(L_B,L_S)}c$ and $\mu_S = \frac{N(L_B,L_S)}{N_S(L_B,L_S)}c$. Also, the first order conditions with respect to L_B and L_S imply $N_B(L_B, L_S) = N_S(L_B, L_S)$. Also by the first order condition with respect to p_b

$$\frac{N(L_B, L_s)}{L_B}\sum\nolimits_s p_s m_{bs}(f_{bs} - v_b - v_s) = \frac{N(L_B, L_S)}{N_B(L_B, L_S)L_B}c - \frac{\gamma_b}{L_B} \tag{92}$$

Because N is homogeneous of degree 1, $N_B(L_B, L_s)L_B + N_S(L_B, L_S)L_S = N(L_B, L_S)$ and since $N_B(L_B, L_S)L_B = N_S(L_B, L_S)L_S$,

$$\frac{N(L_B, L_S)}{N_B(L_B, L_S)L_B} = 2. \tag{93}$$

Consequently

$$\frac{N(L_B, L_S)}{L_B}\sum\nolimits_s p_s m_{bs}(f_{bs} - v_b - v_s) \leq 2c \tag{94}$$

for all b and with equality for $p_b > 0$. This shows that the Constant Surplus Condition is satisfied. Observe the Individual Rationality Condition and Steady State Condition are automatically satisfied. The existence of the Kuhn-Tucker multipliers follows from the same argument as the one provided for Theorem 1 in the appendix.

Proving that any search equilibrium (l, m, v) satisfies the first order and complementary slackness conditions for the planner's problem is analogous to Theorem 1. Observe that Lemma 2 is valid under the matching specification N and so $L = L_B = L_S$ Let v be the multipliers for the steady state constraints and let $\mu_S/L = c/2$, $\mu_B/L = c/2$, $\chi_{bs} = \max\{-s_{bs}, 0\}$, $\omega_{bs} = \max\{s_{bs}, 0\}$, $\gamma_b = c - \frac{1}{2}\sum p_s m_{sb} s_{bs} \geq 0$ and $\gamma_s = c - \frac{1}{2}\sum p_b m_{sb} s_{bs} \geq 0$. Given these definitions, it is straight forward to verify that the first order and complementary slackness conditions are all satisfied.

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MEDS, Kellogg School of Management, Northwestern University, 2001 Sheridan Road, Evanston, IL 60208

E-mail address: a-atakan@kellogg.northwestern.edu