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Strategic Effects of Incomplete and Renegotiation-Proof Contracts*

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Abstract

It is well known that non-renegotiable contracts with third parties may have an effect on the outcome of a strategic interaction and thus serve as a commitment device. We address this issue when contracts are renegotiable. More precisely, we analyze the equilibrium outcomes of two-stage games with renegotiation-proof third-party contracts in relation to the equilibrium outcomes of the same game without contracts. We assume that one of the parties in the contractual relationship is unable to observe everything that happens in the game when played by the other party. This implies that contracts are incomplete and we show that such incompleteness restricts the set of equilibrium outcomes to a subset of Nash equilibrium outcomes of the game without contracts. Introducing renegotiation, in general, imposes further constraints and in some games implies that only subgame perfect equilibrium outcomes can be supported. However, there is a large class of games in which non-subgame perfect equilibrium outcomes can also be supported, and hence, third-party contracts still have strategic implications even when they are renegotiable.

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1 Introduction

As it has been so eloquently illustrated by Schelling (1960), contracts with third parties may have an effect on the outcome of a strategic interaction and therefore could be used as a commitment device. Under the assumption that contracts are *observable and non-renegotiable*, the previous literature has formally illustrated this possibility in many settings. Vickers (1985), Fershtman and Judd (1987), and Sklivas (1987) analyze the effects of managerial compensation contracts on product market competition, and show that such contracts can provide a strategic advantage.[1] Brander and Lewis (1986) do the same for debt contracts, whereas Bolton and Scharfstein (1990) and Snyder (1996) study optimal financial contracts when there is a threat of predation by a "deep-pocket" incumbent. Spencer and Brander (1983), Brander and Spencer (1985), and Eaton and Grossman (1986) study strategic design of trade and industrial policies when firms compete in international markets.

Each one of these models falls into one of two possible categories of games that third-party contracts may induce. In *delegation games*, a player signs a contract that specifies an outcome contingent transfer to an agent, who in turn plays the game in place of the (principal) player. For example, in Fershtman and Judd (1987) the owner of a firm signs a compensation contract with a manager, who in turn chooses the output level in the Cournot game that follows. In *games with side contracts*, the player signs a contract with a third-party but does not delegate the play of the game. In Brander and Lewis (1986), for example, the firm signs a debt contract with a lender and then participates in quantity competition.

Fershtman, Judd, and Kalai (1991), Polo and Tedeschi (2000), and Katz (2006) prove different "folk theorems" for some classes of delegation games under observable and non-renegotiable contracts.[2] The effects of *unobservable and non-renegotiable* third-party contracts are also well-understood. Within the context of delegation games, Katz (1991) showed that the Nash equilibrium outcomes of a game with and without delegation are identical. Koçkesen and Ok (2004) and Koçkesen (2007) addressed the same question within the context of extensive form games and showed that all (and only) Nash equilibrium outcomes of the original game can be supported as a sequential equilibrium outcome of the delegation game. In particular, they showed that outcomes that are not subgame perfect in the original game may arise as a sequential equilibrium outcome of the induced delegation game, i.e., unobservable contracts may have a strategic effect as long as they are non-renegotiable.[3]

Non-renegotiable contracts yield equilibrium outcomes that differ from the subgame perfect equilibrium outcome of the original game by inducing suboptimal behavior (from the perspective of the preferences in the original game) at certain points in the game. These points must be off the equilibrium path, since otherwise the player and the third party could increase the total surplus available to them by inducing optimal play. Therefore, if the game ever reaches such a point, they will have an incentive to renegotiate the existing contract. This implies that, if renegotiation takes place without

[1] Koçkesen, Ok, and Sethi (2000) extend these results to more general classes of games.

[2] Prat and Rustichini (2003) and Jackson and Wilkie (2005) analyze related models in which players can write action contingent contracts before the game is played. However, in Prat and Rustichini (2003) there are multiple principals and agents and principals can contract with any agent, whereas in Jackson and Wilkie (2005) any player can write a contract with any other. Unlike in the literature mentioned in the text, in these papers contractual relationships are not exclusive and the focus is on the efficiency properties of the equilibrium set. Also related is Bhaskar (2008), in which players need to pay a price to a supplier in order to play certain actions that are controlled by this supplier.

[3] Using an example, Katz (1991) also showed that the equivalence between the equilibrium outcomes of games with and without delegation does not hold if one uses refinements of Nash equilibrium. Likewise, Fershtman and Kalai (1997) showed that any outcome of an ultimatum bargaining game can be supported as a trembling hand perfect equilibrium.

any friction, only the subgame perfect equilibrium outcomes of the original game can be supported. In other words, renegotiable third-party contracts have no strategic effect.

Therefore, the question at hand becomes interesting only when there are frictions in the renegotiation process. In this paper we analyze the strategic design of *unobservable and renegotiable* third-party contracts in an environment where such a friction arises quite naturally: We assume that the player who does not participate in the actual play of the game – the principal in delegation games and the third party in games with side contracts – is unable to observe everything that happens in the game.[4] Therefore, contracts can be made contingent only on a partition of the set of outcomes of the underlying game, and are incomplete in this sense. For example, a bank may be able to observe only the level of capacity expansion made by the firm to which it lends, but not those made by this firm's competitors. Similarly, a seller may be able to observe whether an item has been sold by his agent or not, but not the exact price at which the transaction has occurred; a government may observe only the production level of its domestic firm, but not that of the foreign competitor. In these scenarios, the player who actually plays the game may not be able to credibly signal the existence of a mutually beneficial contract and his renegotiation attempt may fail. Motivated by this observation we ask and answer the following question in the current paper: *Which outcomes can be supported in games with unobservable and renegotiable third-party contracts when these contracts are incomplete?*

We limit our analysis to finite two-stage games, in which player 1 moves first by choosing an action $a_1 \in A_1$, and after observing a_1, player 2 chooses an action $a_2 \in A_2$. Let us call this game the *original game*. In the induced *game with third-party contracts*, player 2 and a third party agree on a contract $f : A_2 \to \mathbb{R}$, which specifies a transfer between them as a function of a_2. Note that in delegation games it is the agent who plays the action a_2, whereas in games with side contracts it is player 2. In essence, we assume that the player who does not actually play the original game (the passive player) cannot observe a_1 at any time and hence contracts are incomplete in the sense that they specify a transfer as a function of a_2, rather than (a_1, a_2). The contract is unobservable to player 1, who chooses an action a_1, after which the active player (the agent or player 2) decides whether to end the game by choosing an action a_2 or offer a new contract $g : A_2 \to \mathbb{R}$ to the passive player (player 2 or the third party, respectively). The passive player has to decide whether to accept g or not, without being informed about a_1. Our objective is to characterize the set of outcomes of the original game that can be supported in a perfect Bayesian equilibrium (PBE) of the induced game with third-party contracts.

Therefore, in our setting, where the only friction in the renegotiation process is the inability of the passive player to observe every history, contract incompleteness is a necessary condition for supporting outcomes that are not subgame perfect equilibrium outcomes of the original game. However, contract incompleteness itself brings about interesting issues that are independent of the existence of renegotiation opportunities. Supporting an outcome in a PBE of the game with third-party contracts depends on the ability of writing a contract that gives proper incentives to the active player to play certain strategies. When contracts are complete, as in Koçkesen and Ok (2004) and Koçkesen (2007), finding such contracts is relatively easy, as incentive compatibility does not arise as a binding constraint. When contracts are incomplete, however, only incentive compatible strategies can be supported. We analyze this question in section 5.1 and show that, if payoff functions exhibit increasing differences, then only (and all) the Nash equilibria of the original game in which player 2's strategy

[4] Katz (1991) was the first to consider this scenario within the context of an ultimatum bargaining game and provided the initial motivation for this research.

is increasing can be supported.

As we show in section 5.2, renegotiation imposes further constraints on outcomes that can be supported. In that section, we completely characterize contract-strategy pairs that are renegotiation-proof and give necessary and sufficient conditions for a strategy to be renegotiation-proof. In section 6 we apply our results to an environment that is common to many economically relevant games, such as the Stackelberg and ultimatum bargaining games, and completely characterize the set of outcomes that can be supported with incomplete and renegotiation-proof contracts.

Previous literature has identified two scenarios, which are complementary to ours, in which renegotiable contracts may have a commitment value: (1) games in which there is exogenous asymmetric information between the player and the third party (Dewatripont (1988) and Caillaud, Jullien, and Picard (1995)); and (2) two-stage games with nontransferable utilities (Bensaid and Gary-Bobo (1993)).

Dewatripont (1988) analyzes an entry-deterrence game in which the incumbent signs a contract with a labor union before the game begins. A potential entrant observes the contract and then decides whether to enter or not. Renegotiation takes place after the entry decision is made, during which the union offers a new contract to the incumbent, who has by this time received a payoff relevant private information. The paper shows that commitment effects exist in such a model and may deter entry. This is similar to our model in that the original game is a two-stage game and renegotiation happens after player 1 (the entrant) chooses his action. However, in his model the friction in the renegotiation process arises from an exogenously given asymmetric information, whereas in ours it comes from the inability of the passive player to observe player 1's move. Furthermore, unlike Dewatripont, we analyze arbitrary two-stage games, which enables us to identify conditions on the supportable outcomes in terms of the primitives of the original game.

In Caillaud et al. (1995), unlike in our model, the original game is a simultaneous move game. The game with third-party contracts begins by the player (the principal) offering a publicly observable contract to a third-party (the agent), which may be renegotiated secretly afterwards. After the renegotiation stage, the agent receives a payoff relevant information, after which he may decide to quit. If he does not quit, the agent and the outside party (which is another principal-agent pair) simultaneously choose their actions and the game ends. Their main question is whether publicly announced contracts, which may be secretly renegotiated afterwards, can have a commitment value. They show that the answer to this question depends on whether the original game exhibits strategic complementarity or substitutability and whether there are positive or negative externalities.

Bensaid and Gary-Bobo (1993) also analyze a model in which the original game is a two-stage game and the initial contract can be renegotiated after player 1 chooses an action. However, in their model player 1's action is contractible and observable, but utility is not transferable between player 2 and the third-party. They show that, in a certain class of games, contracts with third parties have a commitment effect, even when they are renegotiable.[5]

Next section presents two simple games, one of which illustrates that non-subgame perfect outcomes can be supported with incomplete and renegotiable contracts, while the other one shows that this is not true in general. Therefore, characterization of equilibrium outcomes that can be supported with such contracts seems to be an interesting matter. Sections 5 and 6 deal with this question in general two-stage games and Section 7 does the same using intuitive criterion (Cho and Kreps (1987)) as

[5]Similarly, Snyder (1996) studies the commitment effects of renegotiable financial contracts in a model with non-transferable utility, where the non-transferability arises from capital market imperfections.

the equilibrium concept. Section 8 concludes with some remarks and open questions, while section 9 contains the proofs of our results.

2 Examples and Motivation

In this section we analyze two simple games, an ultimatum bargaining and a sequential battle-of-the-sexes game, each of which has a unique subgame perfect equilibrium. We will show that renegotiable contracts can support a Nash equilibrium outcome that is not perfect in the bargaining game, while only the subgame perfect equilibrium outcome can be supported in the battle-of-the-sexes game. As we have mentioned in section 1, a game with third-party contracts could take the form of a delegation game or a game with side contracts. To facilitate the exposition of the examples in this section we will use the framework of a game with side contracts. In other words, we will assume that player 2 signs a contract with a third-party before the game begins, and then plays the game herself. Also, we will give all the bargaining power to player 2 in the contractual phase. As it will become apparent later on both of these assumptions are inconsequential for our main results.

ULTIMATUM BARGAINING

Consider a simple ultimatum bargaining game in which player 1 moves first, by choosing the action L or R, after which player 2 moves by choosing l or r. The payoffs corresponding to each outcome are given in the game tree in Figure 1, where the first number is player 1's payoff and the second number player 2's.

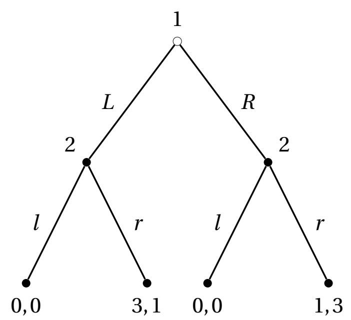

Figure 1: Ultimatum Bargaining Game

The unique subgame perfect equilibrium (SPE) of this game is (L, rr), i.e., player 1 plays L and player 2 plays r after both L and R. There is another Nash equilibrium of this game given by (R, lr). This equilibrium gives player 2 a higher payoff than does the subgame perfect equilibrium, and hence if she could commit to the strategy lr in a credible way she would do so.

Now consider the following game with third-party contracts. Player 2 offers a contract to a neutral third party, which specifies a transfer from player 2 to the third party as a function of the contractible outcomes of the game. The third party may either accept or reject the contract offer. If he accepts, player 1 and player 2 play the game, player 2 receives the game payoff minus the transfer specified by the contract and the third party receives the transfer. If he rejects, then the transfer is zero and the third party receives a fixed payoff δ, while player 1 and 2 receive some small payoff.[6] The set of

[6]Alternatively, we could assume that in case of rejection player 1 and 2 play the game with no contractual obligations. As long as δ is small enough this would not change the analysis that follows.

perfect Bayesian equilibrium outcomes of this game differs depending upon the characteristics of the contracts.

If contracts are observable, non-renegotiable, and complete, in the sense that the transfers can be made conditional on the entire set of outcomes, then the unique PBE outcome of the game is (R,r). A contract that pays the third party δ if the outcome is (L,l) or (R,r) and pays more than $1+\delta$ otherwise is a possible equilibrium contract that achieves this outcome. This is nothing but another illustration of the commitment value of observable, non-renegotiable, and complete contracts.

If contracts are unobservable, then the SPE outcome of the original game, i.e., (L,r), is also an equilibrium outcome of the game, in addition to (R,r). This is an example illustrating the main results in Koçkesen and Ok (2004) and Koçkesen (2007) which state that all Nash equilibrium outcomes can be supported with unobservable (but complete and non-renegotiable) contracts.

If contracts can be renegotiated after the game begins, but they are complete, then the unique equilibrium outcome of the delegation game is the SPE outcome of the original game, irrespective of whether contracts are observable or unobservable. The reason is simple: The only way a non-SPE outcome can be supported is through player 2 playing l after player 1 plays L, which is sequentially irrational from the perspective of player 2's preferences in the original game. Therefore, if player 1 plays L, player 2 and the third party have an incentive to renegotiate the contract so that under the new contract player 2 plays r. In other words, in any PBE, player 2 must play r after any action choice of player 1, and hence player 1 must play L.

The conclusion is entirely different if the third party can observe player 2's action, but not that of player 1. This implies that feasible contracts are incomplete, i.e., they can specify transfers conditional on only player 2's actions but not player 1's actions. We will show that the non-SPE outcome (R,r) is an equilibrium outcome of the induced game with third-party contracts, even if these contracts can be renegotiated.[7] To this end let us specify the renegotiation process as an explicit game form: After player 1 plays, player 2 decides whether to renegotiate by offering a new contract to the third party or not. If she does not offer a contract she chooses an action and the game ends. If she offers a new contract, then the third party either accepts or rejects it, after which player 2 chooses an action and the game ends. If the new contract offer is accepted, then the payoffs are determined according to the new contract, while if rejected, they are determined according to the old contract. The crucial assumption is that the third party cannot observe player 1's action at any time.

The following is a PBE of this game. Player 2 offers the contract f that transfers δ to the third party if she plays r, and transfers $\delta-1$ if she plays l. The third party accepts any contract that gives him an expected payoff of at least δ; player 1's beliefs put probability 1 on f and he plays R; player 2 chooses not to renegotiate f and plays l following L and r following R. In the event of an out-of-equilibrium renegotiation offer after f, the third party believes that player 1 has played R and rejects any contract that transfers him less than δ. Note that in this equilibrium player 2's payoff is $3-\delta$, which implies that as long as $\delta<2$, player 2 prefers to sign such a contract even if she has the option of playing the game without a contract.

Few remarks are in order about this example. First, notice that, in the above equilibrium, player 2 plays l after L, which is not a best response in the original game. Therefore, one may suspect that although the contract specified in the previous paragraph is optimal, it may be weakly dominated by an alternative contract that leads to best response behavior, i.e., playing r, after both L and R. Consider

[7] This has been first observed by Katz (1991) for the ultimatum bargaining game.

such a contract, say g, and note that incentive compatibility implies $g(l) \geq g(r) - 1$ (otherwise player 2 would play l after L). Furthermore, we need to have $g(r) \geq \delta$, for otherwise the third party would reject g and player 2 would obtain some small payoff. Therefore, $g(r) \geq \delta = f(r)$ and $g(l) \geq \delta - 1 = f(l)$, and thus, for any outcome of the game, player 2's payoff after f is at least as large as her payoff after g, which shows that f is not weakly dominated by any contract.[8]

Second, we assumed that the third party accepts any contract that gives him an equilibrium payoff of at least δ. In particular, we allowed the contract to pay him less than δ under some, out-of-equilibrium, circumstances. One might find this unreasonable on the grounds that if player 1 or player 2 makes a mistake in the game that ensues, the third party may end up with a payoff that is smaller than δ, and therefore he would reject such a contract. One way to address this concern is to model the individual rationality constraint of the third party so that he requires a payment of at least δ for every action player 2 might take. This would not change the set of outcomes that can be supported by renegotiable contracts, but may affect how the equilibrium surplus is shared between player 2 and the third party. For example, the least costly such contract that supports the outcome (R, r) would be given by $f(l) = \delta$ and $f(r) = 1 + \delta$, in which case the equilibrium payoff of player 2 would be $2 - \delta$, rather than $3 - \delta$.[9]

Third, in the equilibrium constructed above, the third party believes that player 1 has played R after any out-of-equilibrium renegotiation offer. This might be regarded unreasonable, for there may be contracts that are suboptimal for player 2 to offer after R, but not after L. Therefore, one might want to restrict beliefs to L after such offers. This would be nothing but an application of the intuitive criterion (Cho and Kreps(1987)). It is easy to show that the outcome (R, r) can also be supported in an equilibrium that satisfies the intuitive criterion. More generally, in section 7 we show that all our results go through with minor modifications if we were to adopt this stronger notion of equilibrium.

Fourth, we have to note that unobservability of player 1's actions during the renegotiation phase does not necessarily imply that contracts are incomplete. These actions may be observed, or become verifiable by some other means, at the end of the game, in which case the contracts would in fact be complete. Conversely, even if player 1's actions are observed by both parties at all times, they may be unverifiable, which would render the contracts incomplete. In our analysis above, and in the rest of the paper, we assume that player 1's actions are unobservable by the third party during the game and remain unverifiable throughout. If this were not the case, the outcome (R, r) could not be supported by renegotiable contracts.

SEQUENTIAL BATTLE-OF-THE-SEXES

Consider now the sequential battle-of-the-sexes game given in Figure 2. This game also has a unique SPE, given by (L, lr) and another Nash equilibrium, (R, rr). It can be shown easily that the unique equilibrium outcome of the induced game with third-party contracts is (R, r) if the contracts are observable, non-renegotiable, and complete, whereas the SPE outcome (L, l) can also be supported if contracts are unobservable. If contracts are complete and renegotiable, then only the SPE outcome can be supported. All these observations are in line with those made for the ultimatum bargaining game.

[8] Indeed, Proposition 1 of Fershtman and Kalai (1997) can be adapted to prove that the outcome supported by f, i.e., (R, r), is a trembling hand perfect equilibrium outcome.

[9] Another way would be to transfer some of the bargaining power to the third party so that in equilibrium he receives more than δ. If, for example, $\delta \leq 0.5$, then $f(r) = 1.5$ and $f(l) = 0.5$ would support the same outcome in a way that always

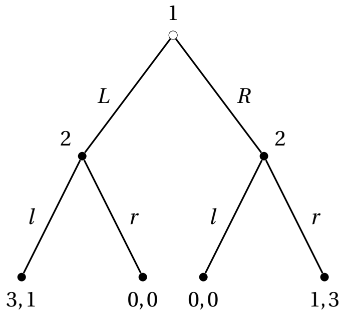

Figure 2: Battle-of-the-Sexes Game

However, the conclusion differs drastically from that in the ultimatum bargaining example if we assume that contracts are renegotiable and incomplete. In this game only the SPE outcome can be supported, while in ultimatum bargaining a non-SPE outcome could also be supported.

Let us prove that the Nash equilibrium outcome (R, r) cannot be supported by renegotiable contracts. Suppose, for contradiction, that there exists a PBE of the delegation game that supports this outcome. Let $f : \{l, r\} \to \mathbb{R}$ be the equilibrium contract that specifies the transfer to be made from player 2 to the third party. For this outcome to be supported, player 2 must be playing r after both actions. Also, in equilibrium, player 2 must extract all the surplus, and hence $f(r) = \delta$. Now consider the renegotiation offer by player 2 given by $g(l) = g(r) = \delta + 0.5$ after player 1 plays L. Note that the third party does not know which action has been played by player 1 when faced with this renegotiation offer. If he accepts g, he will receive a payoff of $\delta + 0.5$ irrespective of player 1's action. If, on the other hand, he rejects it, he believes that player 2 will play r after any action by player 1 and hence he will receive a payoff of δ.[10] Therefore, whatever his beliefs are regarding player 1's action, he has an incentive to accept this renegotiation offer. Furthermore, player 2 has an incentive to make such an offer after player 1 plays L since under f her expected payoff is $-\delta$, whereas under g her expected payoff is $-\delta + 0.5$. This establishes that there is no PBE that supports the outcome (R, r) with renegotiable contracts. Indeed, the unique outcome that can be supported in this case is the SPE outcome of the original game, i.e., (L, l).

In this section we presented two games that are superficially similar but for which delegation with renegotiable contracts gives completely different results. In the rest of the paper we will provide an answer to why this is the case and characterize outcomes that can be supported with renegotiable contracts in arbitrary two-stage extensive form games.

3 The Model

Our analysis starts with a two-player extensive form game, which we call the *original game*. We then allow one of the players to sign a contract with a third party before the game begins and call this new

gives the third party at least δ.

[10] Here, and in the rest of the paper, we restrict player 2's strategy in the game to remain the same if her renegotiation offer is rejected. This is what allows us to state that the third party believes player 2 will play r if he rejects g. Otherwise, outcome (R, r) could also be supported in equilibrium, which would involve a change in player 2's behavior as a result of a failed attempt to change the contract. Since one of our objectives is to identify conditions under which non-subgame perfect outcomes can be supported by renegotiation-proof contracts, we disregard equilibria in which this happens as a result of arbitrary changes in behavior that the mere possibility of renegotiation introduces.

game the *game with third-party contracts.* The contracts specify a transfer between the player and the third party as a function of the contractible outcomes of the original game. After the contract is signed, the game itself may be played by either the third party, in which case we have a delegation game, or by the original player herself, in which case we have a game with side contracts. Although, for the sake of concreteness, we will use the framework of games with side contracts, our main results go through for delegation games as well.

We aim to characterize the equilibrium outcomes of the game with third party contracts in relation with those of the original game. We are particulary interested in whether the induced game with contracts has equilibrium outcomes that are not equilibrium outcomes in the original game, i.e., whether third-party contracts "matter".

As we have seen in the previous section, the nature of the contract space and whether we allow for renegotiation of the contracts during the play of the game is crucial in our query. Previous literature has analyzed this question under the assumption of complete and non-renegotiable contracts, which may be observable or unobservable by outside parties. Our focus, in contrast, is on contracts that can be renegotiated at any point in a costless and secret manner. This immediately implies that contracts are unobservable, since they can be renegotiated before the game begins. If we also assume that contracts are complete, i.e., the contractible outcomes are all the outcomes of the original game, and there are no frictions in the renegotiation process, such as asymmetric information between the contracting parties, third-party contracts cannot "alter" the set of equilibrium outcomes of the original game. Therefore, one has to introduce some sort of friction into the renegotiation process to make the analysis interesting.

We analyze a model in which the friction arises from the assumption that one of the contracting parties cannot observe all the histories of the original game when it is played by the other party. In other words, we assume that in games with side contracts, the third party, and in delegation games the principal, is unable to observe everything that happens in the game. We believe that this is a natural environment to consider. For example, the principal may loose the ability to perfectly monitor the play of the game once she delegates the play to an agent, or a bank may not be able to observe everything that matters to the firm it lends to. In any case, we think that the resulting model is quite rich and introduces new dimensions into the analysis of contracts in strategic settings, such as incompleteness and moral hazard.

The assumption that one of the contracting parties cannot observe every history in the game implies that monetary transfers cannot be conditioned on every terminal history of the game and hence contracts must be incomplete.[11] Furthermore, if contracting parties cannot observe each other's actions perfectly, then moral hazard becomes an issue in contract design. In this paper we focus on incompleteness, leaving the analysis of issues associated with moral hazard to future work.

Contract incompleteness in our setting, therefore, is a necessary condition for obtaining non-trivial results regarding the effects of renegotiation. However, incompleteness introduces novel issues into the analysis and is interesting in itself. The set of equilibrium outcomes that can be attained in games with third-party contracts depends on the ability of the contracts to give the right incentives to play certain actions. Incentive compatibility is satisfied in a trivial way in models with complete contracts (such as the one in Koçkesen and Ok (2004)). However, as we will see later on, incentive compatibility becomes a binding constraint in a model with incomplete contracts and obtaining sharp

[11] As we noted before, we assume that player 1's actions remain unverifiable throughout the game.

results requires imposing further structure on the model, such as assuming that payoff functions exhibit increasing differences.

The main intuition behind our results is best seen in a simple model in which the original game has only two stages: Player 1 moves first and player 2 second. Limiting player 1's move to only the first stage makes formulating the model, e.g., introducing an order structure on the set of histories in the game and defining increasing differences, much easier and renders the results more transparent. For these reasons, we restrict the analysis to two-stage games.

Limiting the analysis to two-stage games simplifies the analysis further as we may, without loss of generality, assume that only the second mover can sign contracts with third parties. Third-party contracts introduce equilibrium outcomes that are not equilibrium outcomes in the original game by inducing sequentially irrational play (from the perspective of the preferences in the original game) at information sets that are not reached in equilibrium. Since player 1 moves only once, at the beginning of the game, allowing him to delegate would not change the set of equilibrium outcomes at all.[12]

In light of these observations, we define the *original game*, denoted G, as a two-player finite extensive form game with perfect information. We assume that this game is composed of two stages: Player 1 chooses $a_1 \in A_1$, and player 2, after observing a_1, chooses $a_2 \in A_2$, where A_1 and A_2 are finite sets. Payoff function of player $i \in \{1,2\}$ is given by $u_i : A \to \mathbb{R}$, where $A = A_1 \times A_2$.

The *game with incomplete and non-renegotiable third-party contracts*, denoted $\Gamma(G)$, is a three player extensive form game described by the following sequence of events:

Stage I. Player 2 offers a contract $f : A_2 \to \mathbb{R}$ to a third party.

Stage II. The third party accepts (denoted y) or rejects (denoted n) the contract.

1. In case of rejection the game ends, the third party receives a fixed payoff of $\delta \in \mathbb{R}$, and player 1 and 2 receive $-\infty$.[13]
2. In case of acceptance, the game goes to Stage III.

Stage III. Player 1 chooses an action $a_1 \in A_1$ (without observing the contract), player 2 observes a_1.

Stage IV. Player 2 chooses an action $a_2 \in A_2$.

Since we assume that if the contract offer is rejected, the game ends and players 1 and 2 receive very small payoffs, the contract offer is accepted in all equilibria. Therefore, we will, for the sake of notational simplicity, denote the set of outcomes as $Z = \mathscr{C} \times A$, where $\mathscr{C} = \mathbb{R}^{A_2}$ is the contract space. For any outcome $(f, a) \in Z$ the payoff functions are given by

$$\begin{aligned}
v_1(f,a) &= u_1(a_1,a_2) \\
v_2(f,a) &= u_2(a_1,a_2) - f(a_2) \\
v_3(f,a) &= f(a_2)
\end{aligned}$$

where v_3 is the payoff function of the third party.

[12] Of course, as it was shown in Koçkesen (2007), in games with more than two stages this is not the case.

[13] Alternatively, we could assume that if the third party rejects an offer, then the original game is played without a contract. However, this assumption introduces additional notation and technical details without changing our results in any substantive way.

The above formulation assumes that after the contract is signed, it is player 2 who actually plays the game, i.e., we have a game with side contracts. In a delegation game, it is the third party who plays the game, in which case player 2's payoff function would be given by $f(a_2)$, and the third party's by $u_2(a_1, a_2) - f(a_2)$.[14] As we have indicated before, all our results go through without modification irrespective of who plays the game, but for ease of exposition we assume that the game is with side contracts.

We also assume that player 2 has the entire bargaining power during the contracting phase. This assumption has no effect on our results regarding the set of equilibrium outcomes, but clearly has implications regarding the equilibrium payoff of player 2. Also note that δ could represent either the outside option of the third party, such as that of an agent or a lender, or could be used to model some other constraint on the transfers, such as the upper bound on the amount of export subsidy.

The game is with renegotiable contracts if the contracting parties can renegotiate the contract after Stage III and before Stage IV. We assume that renegotiation can be initiated only by the player who actually plays the game. However, as it will become apparent after we introduce our concept of renegotiation-proofness, the results remain intact if the renegotiation process is initiated by the other player. The following sequence of events describe the renegotiation process after any history (f, a_1).

Stage III(i). Player 2 either offers a new contract $g \in \mathscr{C}$ to the third party or chooses an action a_2. In the latter case the game ends and the outcome is (f, a).

Stage III(ii). If player 2 offers a new contract, the third party (without observing a_1) either accepts (denoted y) or rejects (denoted n) the offer.

If the third party rejects the renegotiation offer g, then player 2 chooses $a_2 \in A_2$ and the outcome is payoff equivalent to (f, a). If he accepts, then player 2 chooses $a_2 \in A_2$ and the outcome is payoff equivalent to (g, a). This completes the description of the *delegation game with incomplete and renegotiable contracts*, which we denote as $\Gamma_R(G)$.

A *behavior strategy* for player $i \in \{1, 2, 3\}$ is defined as a set of probability measures $\beta_i \equiv \{\beta_i[I] : I \in \mathscr{I}_i\}$, where $\mathscr{I}_i$ is the set of information sets of player i and $\beta_i[I]$ is defined on the set of actions available at information set I. One may write $\beta_i[h]$ for $\beta_i[I]$ for any history $h \in I$. By a *system of beliefs*, we mean a set $\mu \equiv \{\mu[I] : I \in \mathscr{I}_i$ for some $i\}$, where $\mu[I]$ is a probability measure on I. A pair (β, μ) is called an *assessment*. An assessment (β, μ) is said to be a *perfect Bayesian equilibrium* (PBE) if (1) each player's strategy is optimal at every information set given her beliefs and the other players' strategies; and (2) beliefs at every information set are consistent with observed histories and strategies.[15]

4 The Query

We will limit our analysis to pure behavior strategies, and hence a strategy profile of the original game G is given by $(b_1, b_2) \in A_1 \times A_2^{A_1}$. For any behavior strategy profile (b_1, b_2) in G, we say that an assessment (β, μ) in $\Gamma(G)$ *induces* (b_1, b_2) if in $\Gamma(G)$ player 1 plays according to b_1 and, after the equilibrium

[14] In a delegation game, this payoff specification would be reasonable if the third party can inherit player 2's preferences once the game is delegated to him. Consider, for example, a seller who delegates the sale of an item to an agent and suppose that she cannot observe the actual price at which the item is sold. In this case the contract would specify a payment from the agent to the seller contingent upon whether a sale has occurred or not. If the seller and the agent care only about money, then the above payoff specification would indeed be the appropriate one.

[15] See Fudenberg and Tirole (1991) for a precise definition of perfect Bayesian equilibrium.

contract, player 2 plays according to b_2. Note that in $\Gamma_R(G)$, player 2 may choose an action $a_2 \in A_2$ either without renegotiating the initial contract or after attempting renegotiation. Therefore, an assessment (β, μ) in $\Gamma_R(G)$ may induce a behavior strategy profile (b_1, b_2) in G in these two different ways.

We restrict our attention to equilibria in which the equilibrium contract is not renegotiated. As Beaudry and Poitevin (1995) point out, this is necessary for renegotiation to have any bite, as one can always replicate an equilibrium outcome of the game without renegotiation by making player 2 offer an initial contract that is accepted only because it is going to be renegotiated later on.[16] This leads to the following definition.

Definition 1 (Renegotiation-Proof Equilibria)**.** A perfect Bayesian equilibrium (β^*, μ^*) of $\Gamma_R(G)$ is *renegotiation-proof* if the equilibrium contract is not renegotiated after any a_1.

Note that the set of renegotiation-proof equilibria is actually a subset of perfect Bayesian equilibria in which the equilibrium contract is not renegotiated. The latter would be defined so that the equilibrium contract is not renegotiated after any action of player 1 that gives him a higher payoff under a renegotiated contract than the equilibrium payoff. However, working with this weaker notion of renegotiation-proofness would only introduce additional complexity into our presentation without changing the main results in any interesting way.

Definition 2. A strategy profile (b_1, b_2) of the original game G can be *supported with incomplete and non-renegotiable contracts* if there exists a perfect Bayesian equilibrium of $\Gamma(G)$ that induces (b_1, b_2). Similarly, a strategy profile (b_1, b_2) of the original game G can be *supported with incomplete and renegotiable contracts* if there exists a renegotiation-proof perfect Bayesian equilibrium of $\Gamma_R(G)$ that induces (b_1, b_2).

Our main query can therefore be phrased as follows:

> *Which outcomes of a given original game can be supported with incomplete and renegotiable (or non-renegotiable) contracts?*

Clearly, in our model, if an outcome can be supported with renegotiable contracts, it can also be supported with non-renegotiable contracts.[17] Therefore, we start by characterizing the set of outcomes that can be supported with non-renegotiable contracts before we analyze the restrictions imposed by renegotiation. We should emphasize that $\Gamma(G)$ is with unobservable but *incomplete* contracts. The results provided in Koçkesen and Ok (2004) are valid only for games with *complete contracts* and hence do not provide the relevant starting point for our analysis. Applied to our setting, Koçkesen and Ok (2004) implies that every Nash equilibrium outcome can be supported with complete contracts, whereas, as we will see in the next section, only a subset of these can be supported when the contracts are incomplete.

5 Main Results

In this section we will provide an answer to our main query, first for incomplete and non-renegotiable contracts and then for renegotiable contracts.

[16] See also Maskin and Tirole (1992) on this point.

[17] This claim is proved as (the [Only if]) part of Proposition 2 in Section 9.

5.1 Incomplete and non-Renegotiable Contracts

Let G be an arbitrary original game and $\Gamma(G)$ be the game with incomplete and non-renegotiable third-party contracts. We first prove the following.

Proposition 1. *A strategy profile (b_1^*, b_2^*) of G can be supported with incomplete and non-renegotiable contracts if and only if*

1. *(b_1^*, b_2^*) is a Nash equilibrium of G*

   and there exists an $f \in \mathscr{C}$ such that

2. *$f(b_2^*(b_1^*)) = \delta$,*

3. *$u_2(a_1, b_2^*(a_1)) - f(b_2^*(a_1)) \geq u_2(a_1, b_2^*(a_1')) - f(b_2^*(a_1'))$, for all $a_1, a_1' \in A_1$.*

Proposition 1 provides necessary and sufficient conditions for an outcome of an arbitrary original game to be supported with incomplete and non-renegotiable contracts. Condition 1 states that only Nash equilibrium outcomes can be supported, which, as in Koçkesen and Ok (2004), follows from sequential rationality of players 1 and 2. Condition 2 simply states that the third party does not receive rents in equilibrium, whereas condition 3 is the incentive compatibility constraint imposed by the incompleteness of contracts.

Although Proposition 1 provides a complete characterization, it falls short of precisely identifying the supportable outcomes in terms of the primitives of the original game. As it is standard in adverse selection models, we can obtain a much sharper characterization if we impose an order structure on A_1 and A_2 and assume that player 2's payoff function v_2 exhibits increasing differences. Given the definition of v_2, this is equivalent to assuming that u_2 has increasing differences. To this end, let $\succsim_1$ be a linear order on A_1 and $\succsim_2$ a linear order on A_2, and denote their asymmetric parts by $\succ_1$ and $\succ_2$, respectively.

Definition 3 (Increasing Differences)**.** $u_2 : A_1 \times A_2 \to \mathbb{R}$ is said to have *increasing differences* in $(\succsim_1, \succsim_2)$ if $a_1 \succsim_1 a_1'$ and $a_2 \succsim_2 a_2'$ imply that $u_2(a_1, a_2) - u_2(a_1, a_2') \geq u_2(a_1', a_2) - u_2(a_1', a_2')$. It is said to have *strictly increasing differences* if $a_1 \succ_1 a_1'$ and $a_2 \succ_2 a_2'$ imply that $u_2(a_1, a_2) - u_2(a_1, a_2') > u_2(a_1', a_2) - u_2(a_1', a_2')$.

Definition 4 (Increasing Strategies)**.** $b_2 : A_1 \to A_2$ is called *increasing* in $(\succsim_1, \succsim_2)$ if $a_1 \succsim_1 a_1'$ implies that $b_2(a_1) \succsim_2 b_2(a_1')$.

From now on, we restrict our analysis to games in which there exists a linear order $\succsim_1$ on A_1 and a linear order $\succsim_2$ on A_2 such that u_2 has strictly increasing differences in $(\succsim_1, \succsim_2)$. We then have the following result.

Theorem 1. *A strategy profile (b_1^*, b_2^*) of G can be supported with incomplete and non-renegotiable contracts if and only if (b_1^*, b_2^*) is a Nash equilibrium of G and b_2^* is increasing.*

This result completely characterizes the strategy profiles that can be supported with incomplete contracts and precisely identifies the restrictions imposed by incompleteness. While earlier papers showed that any Nash equilibrium of the original game can be supported by unobservable and *complete* contracts, this result shows that only the subset of Nash equilibria in which the second player plays an increasing strategy can be supported if, instead, contracts are incomplete.

The reason why only increasing strategies of the second player can be supported is very similar to the reason why only increasing strategies can be supported in standard adverse selection models: If the payoff function of player 2 exhibits increasing differences, then incentive compatibility is equivalent to increasing strategies. The set of actions of player 1, A_1, plays the role of the type set of the agent in standard principal-agent models. The fact that contracts cannot be conditioned on A_1 transforms the model into an adverse selection model, which, combined with increasing differences exhibited by $u_2(a_1, a_2) - f(a_2)$, necessitates increasing strategies to satisfy incentive compatibility, i.e., condition 3 of Proposition 1. We prove sufficiency by using a theorem of the alternative.

We now move on to analyze the effects of renegotiable contracts. As we noted before, if contracts are renegotiable and complete, then the only equilibrium that can be supported is the subgame perfect equilibrium of the original game. Therefore, for renegotiable contracts to have any effect on the outcome of the game, they must be incomplete. However, as we have just seen, contract incompleteness also acts as a restriction on the set of supportable outcomes. Therefore, our query to identify outcomes that can be supported with *renegotiable and incomplete contracts* seems to be interesting. The next section attacks precisely this problem.

5.2 Incomplete and Renegotiable Contracts

Let G be an arbitrary original game and $\Gamma_R(G)$ be the induced game with incomplete and renegotiable third-party contracts. As stated before we would like to identify the set of outcomes of G that can be supported by renegotiation-proof perfect Bayesian equilibria of $\Gamma_R(G)$.

When faced with a renegotiation offer, the third party has to form beliefs regarding how player 2 would play under the new contract and compare his payoffs from the old and the new contracts to decide whether to accept it or not. As we have seen in section 5.1, contract incompleteness imposes incentive compatibility constraints on the strategy of player 2, and therefore the third party has to restrict his beliefs to strategies that are incentive compatible under the new contract. For future reference, let us first define incentive compatibility as a property of any contract-strategy pair $(f, b_2) \in \mathscr{C} \times A_2^{A_1}$.

Definition 5 (Incentive Compatibility)**.** $(f, b_2) \in \mathscr{C} \times A_2^{A_1}$ is *incentive compatible* if

$$u_2(a_1, b_2(a_1)) - f(b_2(a_1)) \geq u_2(a_1, b_2(a_1')) - f(b_2(a_1')) \text{ for all } a_1, a_1' \in A_1.$$

To understand the constraints imposed by renegotiation-proofness suppose that (β, μ) is a renegotiation-proof PBE of $\Gamma_R(G)$ and let f be the equilibrium contract and b_2^* be the equilibrium strategy of player 2 following f. Now suppose that for a particular choice of action by player 1, say a_1', there exists an incentive compatible contract-strategy pair (g, b_2) such that $u_2(a_1', b_2(a_1')) - g(b_2(a_1')) > u_2(a_1', b_2^*(a_1')) - f(b_2^*(a_1'))$ and $g(b_2(a_1)) > f(b_2^*(a_1))$ for all a_1. This implies that, after a_1' is played, player 2 will have an incentive to renegotiate and offer g and the third party will have an incentive to accept it. This would contradict that (β, μ) is a renegotiation-proof PBE of $\Gamma_R(G)$. This leads to the following definition.

Definition 6 (Renegotiation-Proofness)**.** We say that $(f, b_2^*) \in \mathscr{C} \times A_2^{A_1}$ is *renegotiation-proof* if for all $a_1 \in A_1$ for which there exists an incentive compatible $(g, b_2) \in \mathscr{C} \times A_2^{A_1}$ such that

$$u_2(a_1, b_2(a_1)) - g(b_2(a_1)) > u_2(a_1, b_2^*(a_1)) - f(b_2^*(a_1)) \tag{1}$$

and

$$g(b_2(a_1)) > f(b_2^*(a_1)) \tag{2}$$

there exists an $a_1' \in A_1$ such that

$$f(b_2^*(a_1')) \geq g(b_2(a_1')) \tag{3}$$

Again, the intuition behind this definition is clear: Whenever there is an a_1 after which there is a contract g and an incentive compatible continuation play b_2 such that the contracting parties both prefer g over f (i.e., (1) and (2) hold), there exists a belief of the third party under which it is optimal to reject g, which is implied by (3).[18]

In a similar vein, we have the following definition for a renegotiation-proof strategy.

Definition 7 (Renegotiation-Proof Strategy)**.** A strategy $b_2 \in A_2^{A_1}$ is *renegotiation-proof* if there exists an $f \in \mathscr{C}$ such that (f, b_2) is incentive compatible and renegotiation-proof.

Intuitively, Definition 7 seems to identify the conditions that b_2 must satisfy to be induced by a renegotiation-proof perfect Bayesian equilibrium of $\Gamma_R(G)$. The following result proves that this intuition is correct.

Proposition 2. *A strategy profile (b_1^*, b_2^*) of G can be supported with incomplete and renegotiable contracts if and only if (b_1^*, b_2^*) is a Nash equilibrium of G and b_2^* is increasing and renegotiation-proof.*

Unfortunately, it is difficult to apply Definitions 6 and 7 directly to an arbitrary game to ascertain the restrictions that renegotiation-proofness imposes on contracts and strategies. However, the conditions themselves are all linear inequalities and we can use theorems of the alternative to understand these restrictions better in terms of the primitives of the original game. To this end, let the number of elements in A_1 be equal to n and order its elements so that $a_1^n \succsim_1 a_1^{n-1} \succsim_1 \cdots a_1^2 \succsim_1 a_1^1$. For any contract-strategy pair (f, b_2), define $f_j = f(b_2(a_1^j))$, $j = 1, \ldots, n$, and let, with an abuse of notation, $f \in \mathbb{R}^n$ be the vector whose jth component is given by f_j.

First, note that, under increasing differences, incentive compatibility of (g, b_2) is equivalent to b_2 being increasing and conditions (1) and (2) imply that $u_2(a_1, b_2(a_1)) > u_2(a_1, b_2^*(a_1))$. In other words, condition (3) needs to be satisfied for every $a_1^i, i = 1, \ldots, n$, and increasing strategy that leads to a higher surplus for the contracting parties. For a given i, let us define the set of all such strategies as

$$\mathfrak{B}(i, b_2^*) = \{b_2 \in A_2^{A_1} : b_2 \text{ is increasing and } u_2(a_1^i, b_2(a_1^i)) > u_2(a_1^i, b_2^*(a_1^i))\}. \tag{4}$$

Second, by Definition 6, (f, b_2^*) is *not* renegotiation-proof if and only if there exist i and incentive compatible (g, b_2) such that $u_2(a_1^i, b_2(a_1^i)) - g_i > u_2(a_1^i, b_2^*(a_1^i)) - f_i$ and $g_j > f_j$ for all j. When u_2 has increasing differences, incentive compatibility of (g, b_2) is equivalent to the local upward and downward constraints:

$$g_j - g_{j+1} \leq u_2(a_1^j, b_2(a_1^j)) - u_2(a_1^j, b_2(a_1^{j+1})), \quad j = 1, \ldots, n-1$$
$$-g_{j-1} + g_j \leq u_2(a_1^j, b_2(a_1^j)) - u_2(a_1^j, b_2(a_1^{j-1})), \quad j = 2, \ldots, n$$

[18] One may find this definition too weak as it allows the beliefs to be arbitrary following an off-the-equilibrium renegotiation offer. A more reasonable alternative could be to require the beliefs to satisfy intuitive criterion. In Section 7 we show that our results go through with minor modifications when we adopt this stronger version of renegotiation-proofness.

We can write these inequalities in matrix form as $Dg \le U(b_2)$, where D is a matrix of coefficients and $U(b_2)$ a column vector with $2(n-1)$ components, whose component $2j-1$ is given by

$$U(b_2)_{2j-1} = u_2(a_1^j, b_2(a_1^j)) - u_2(a_1^j, b_2(a_1^{j+1}))$$

and component $2j$ is given by

$$U(b_2)_{2j} = u_2(a_1^{j+1}, b_2(a_1^{j+1})) - u_2(a_1^{j+1}, b_2(a_1^j))$$

Therefore, (f, b_2^*) is not renegotiation-proof if and only if there exist i, b_2, and $\varepsilon \in \mathbb{R}^n$ such that

$$D(f+\varepsilon) \le U(b_2), \quad \varepsilon_i < u_2(a_1^i, b_2(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i)), \quad \varepsilon \gg 0$$

These conditions can be written as $[Ax \gg 0, Cx \ge 0$ has a solution $x]$, once the vector x and matrices A and C are appropriately defined. Motzkin's theorem of the alternative (stated as Lemma 3 in section 9) then implies that the necessary and sufficient condition for being renegotiation-proof is $[A'y_1 + C'y_2 = 0, y_1 > 0, y_2 \ge 0$ has a solution $y_1, y_2]$ (See Lemma 4 in section 9). The fact that u_2 has increasing differences can then be used to prove the equivalence of this condition to the one stated in the following theorem.

Theorem 2. (f, b_2^*) *is renegotiation-proof if and only if for any* $i \in \{1, 2, \dots, n\}$ *and* $b_2 \in \mathfrak{B}(i, b_2^*)$ *there exists a* $k \in \{1, 2, \dots, i-1\}$ *such that*

$$u_2(a_1^i, b_2(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i)) + \sum_{j=k}^{i-1} U(b_2)_{2j-1} \le f_k - f_i \tag{5}$$

or there exists an $l \in \{i+1, i+2, \dots, n\}$ *such that*

$$u_2(a_1^i, b_2(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i)) + \sum_{j=i+1}^{l} U(b_2)_{2(j-1)} \le f_l - f_i \tag{6}$$

In order to apply this theorem directly to a given game and a strategy b_2^* one would first identify the set of contracts under which player 2 has an incentive to play b_2^*, and then check if any of those contracts satisfies the conditions of the theorem. It is best to illustrate this using the examples introduced in Section 2. For both the ultimatum bargaining and sequential battle-of-the-sexes games, define $\precsim_1$ and $\precsim_2$ so that $R \succ_1 L$ and $r \succ_2 l$ and note that u_2 has strictly increasing differences in $(\precsim_1, \precsim_2)$.

Ultimatum Bargaining

There are three Nash equilibria of the game: (L, rl), (L, rr), and (R, lr). The second one is the unique SPE and it has the same outcome as the first. The third one is not subgame perfect. Notice that the last two equilibria have increasing b_2 and hence, by Theorem 1, can be supported with incomplete and non-renegotiable contracts. Since the SPE can be supported with renegotiable contracts as well, the question is whether (R, lr) can be supported with incomplete and renegotiable contracts.[19]

Any equilibrium contract f that supports (R, lr) must satisfy the incentive compatibility constraint given by $1 \le f(r) - f(l) \le 3$. Since player 2 is already best responding after R, a Pareto im-

[19]Clearly, if a contract supports a SPE, it is renegotiation-proof as there is no $a_1 \in A_1$ such that (1) and (2) hold.

proving renegotiation can happen only after L and it must lead to $b_2(L) = r$. Incentive compatibility implies that b_2 is increasing, and therefore, $b_2(R) = r$. From Theorem 2, (f, b_2^*) is renegotiation-proof if and only if

$$[u_2(L, b_2(L)) - u_2(L, b_2^*(L))] + [u_2(R, b_2(R)) - u_2(R, b_2(L))] \leq f(b_2^*(R)) - f(b_2^*(L))$$

Substituting for b_2^* and b_2, this is equivalent to $1 \leq f(r) - f(l)$. Since incentive compatibility holds if $1 \leq f(r) - f(l) \leq 3$, we conclude that $b_2^* = lr$ can be supported with a renegotiation-proof contract and hence (R, lr) can be supported with incomplete and renegotiable contracts.

SEQUENTIAL BATTLE-OF-THE-SEXES

There are three Nash equilibria of the game: (L, ll), (L, lr), and (R, rr). The second one is the unique SPE and it has the same outcome as the first. The third one is not subgame perfect. All of these equilibria have an increasing b_2 and hence can be supported with incomplete and non-renegotiable contracts. The question again is whether the (non-subgame perfect) Nash equilibrium (R, rr) can be supported with incomplete and renegotiable contracts.

The only possibility for a Pareto improving renegotiation is after L and it must induce $b_2(L) = l$. Theorem 2 implies that if (f, b_2^*) is renegotiation-proof then

$$[u_2(L, b_2(L)) - u_2(L, b_2^*(L))] + [u_2(R, b_2(R)) - u_2(R, b_2(L))] \leq f(b_2^*(R)) - f(b_2^*(L))$$

or $u_2(R, b_2(R)) + 1 \leq 0$, which is impossible since $u_2(R, b_2(R)) \geq 0$. We conclude that it is not possible to support (R, rr) with incomplete and renegotiable contracts.

Although Theorem 2 is quite powerful in applications, it would still be desirable to obtain general results that involve only the primitives of the original game. In particular, we would like to obtain conditions for a strategy b_2^* to be supportable with incomplete and renegotiable contracts. Given Proposition 2, this requires identifying renegotiation-proof strategies, i.e., those for which there exists an $f \in \mathscr{C}$ such that (f, b_2^*) is incentive compatible and renegotiation-proof. For any $k, i \in \{1, \ldots, n\}$, incentive compatibility implies

$$f_k - f_i \leq u_2(a_1^k, b_2^*(a_1^k)) - u_2(a_1^i, b_2^*(a_1^i)).$$

Together with Theorem 2, we then have the following necessary condition for b_2^* being renegotiation-proof: For any $i = 1, \ldots, n$ and $b_2^i \in \mathfrak{B}(i, b_2^*)$ there exists a $k \in \{1, 2, \ldots, i-1\}$ such that

$$u_2(a_1^i, b_2(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i)) + \sum_{j=k}^{i-1} U(b_2)_{2j-1} \leq u_2(a_1^k, b_2^*(a_1^k)) - u_2(a_1^i, b_2^*(a_1^i))$$

or there exists an $l \in \{i+1, i+2, \ldots, n\}$ such that

$$u_2(a_1^i, b_2(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i)) + \sum_{j=i+1}^{l} U(b_2)_{2(j-1)} \leq u_2(a_1^l, b_2^*(a_1^l)) - u_2(a_1^i, b_2^*(a_1^i))$$

In fact, again utilizing a theorem of the alternative (Gale's theorem of inequalities), we can make this condition tighter. To facilitate the exposition, we first introduce the following definition.

Definition 8. For any $i = 1, \ldots, n$ and $b_2 \in \mathfrak{B}(i, b_2^*)$ we say that $m(b_2) \in \{1, 2, \ldots, n\}$ is a *blocking action* if

$$u_2(a_1^i, b_2(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i)) + \sum_{j=m(b_2)}^{i-1} U(b_2)_{2j-1} \leq \sum_{j=m(b_2)}^{i-1} U(b_2^*)_{2j-1} \tag{7}$$

or

$$u_2(a_1^i, b_2(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i)) + \sum_{j=i+1}^{m(b_2)} U(b_2)_{2(j-1)} \leq \sum_{j=i+1}^{m(b_2)} U(b_2^*)_{2(j-1)} \tag{8}$$

We then obtain the following result.

Proposition 3. *A strategy $b_2^* \in A_2^{A_1}$ is renegotiation-proof only if for any $i = 1, \ldots, n$ and $b_2^i \in \mathfrak{B}(i, b_2^*)$ there is a blocking action.*[20]

However, this condition is not sufficient for renegotiation-proofness and becomes sufficient with an additional condition on the blocking actions for different a_1's. More precisely,

Proposition 4. *A strategy $b_2^* \in A_2^{A_1}$ is renegotiation-proof if for any $i = 1, \ldots, n$ and $b_2^i \in \mathfrak{B}(i, b_2^*)$ there is a blocking action $m(b_2^i)$ such that $k < l$, $m(b_2^k) > k$, and $m(b_2^l) < l$ imply $m(b_2^k) \leq m(b_2^l)$.*

The conditions given in Propositions 3 and 4 coincide when player 1 has only two actions. Therefore, Proposition 3 is a full characterization result for such games, including our running examples. Let us use this proposition to show that $b_2^* = lr$ is renegotiation-proof in ultimatum bargaining example. Let $L = a_1^1$ and $R = a_1^2$ and note that $\mathfrak{B}(1, b_2^*) = \{rr\}$ and $\mathfrak{B}(2, b_2^*) = \emptyset$. Therefore, we only need to check if there is a blocking action for $i = 1$ and $b_2 = rr$, the only candidate for which is R. Applying (8), we get

$$u_2(L, r) - u_2(L, l) + u_2(R, r) - u_2(R, r) \leq u_2(R, r) - u_2(R, l)$$

which is satisfied. We therefore conclude that $b_2^* = lr$ is renegotiation proof.

Now let us show that $b_2^* = rr$ is not renegotiation-proof in the battle-of-the-sexes game. In this case, $\mathfrak{B}(2, b_2^*) = \emptyset$ and $\mathfrak{B}(1, b_2^*) = \{ll, lr\}$. It is sufficient to show that there is no blocking action for $i = 1$ and $b_2 = lr$. The only candidate for a blocking action is R and we need the following inequality to be satisfied

$$u_2(L, l) - u_2(L, r) + u_2(R, r) - u_2(R, l) \leq u_2(R, r) - u_2(R, r).$$

Obviously, this is not true and we conclude that $b_2^* = rr$ is not renegotiation-proof.

When A_1 has more than two actions the condition stated in Proposition 3 is not sufficient anymore and obtaining a full characterization for such games requires introducing more structure into the model. In the next section we do this for a large class of economically relevant games.

6 A Special Environment and Applications

In this section we analyze a class of games that includes many economic models, among which are Stackelberg and entry games, sequential Bertrand games with differentiated products, and ultimatum bargaining. To define this class of games, take any original game G and consider the strategic form game $S(G) = (\{1, 2\}, (A_1, A_2), (u_1, u_2))$, i.e., $S(G)$ is the simultaneous move version of G. Let br_i denote

[20] The fact that this is a tighter condition follows from an easy induction argument that establishes $\sum_{j=k}^{i-1} U(b_2^*)_{2j-1} \leq u_2(a_1^k, b_2^*(a_1^k)) - u_2(a_1^i, b_2^*(a_1^i))$.

a selection from the best-response correspondence of player i in $S(G)$, i.e., $br_i(a_{-i}) \in BR_i(a_{-i})$ for all $a_{-i} \in A_{-i}$. Also, let $NE(S(G))$ be the set of pure strategy Nash equilibria of $S(G)$ and denote the smallest and the largest (in $\succsim_1$) pure strategy Nash equilibrium action of player 1 by $\underline{a}_1^{NE}$ and $\overline{a}_1^{NE}$, respectively.

Definition 9. u_1 has *positive externality* in $\succsim_2$ if $a_2 \succsim_2 a_2'$ implies $u_1(a_1, a_2) \geq u_1(a_1, a_2')$ for all $a_1 \in A_1$.

Definition 10. u_1 is *single-peaked* in $\succsim_1$ if for all $br_1 \in BR_1$ and $a_2 \in A_2$, $br_1(a_2) \succsim_1 a_1' \succsim_1 a_1$ implies $u_1(a_1', a_2) \geq u_1(a_1, a_2)$ and $a_1 \succsim_1 a_1' \succsim_1 br_1(a_2)$ implies $u_1(a_1', a_2) \geq u_1(a_1, a_2)$. Define single-peaked u_2 in a similar manner.

Definition 11. For any original game G, we say that $NE(S(G))$ is *stable* if, for any selection (br_1, br_2), $a_1 \succsim_1 \overline{a}_1^{NE}$ implies $a_1 \succsim_1 br_1(br_2(a_1))$ and $a_1 \precsim_1 \underline{a}_1^{NE}$ implies $a_1 \precsim_1 br_1(br_2(a_1))$.

Positive externality and single-peakedness are standard conditions. Stability condition, on the other hand, is necessary for the convergence of best response dynamics to the interval $[\underline{a}_1^{NE}, \overline{a}_1^{NE}]$. Indeed, if there exist $a_1^0 \succ_1 \overline{a}_1^{NE}$ and (br_1, br_2) such that $a_1^0 \prec_1 br_1(br_2(a_1^0))$, then starting from $(a_1^0, br_2(a_1^0))$, the dynamics defined by $(a_1^{t+1}, a_2^{t+1}) = (br_1(a_2^t), br_2(a_1^t))$, $t = 0, 1, \ldots$, would be such that $a_1^t \succ_1 \overline{a}_1^{NE}$ for all $t = 0, 1, \ldots$.

Let $\mathscr{G}$ denote the class of games G in which u_1 and u_2 are single-peaked, u_2 has strictly increasing differences in $(\succsim_1, \succsim_2)$, u_1 has positive externality, and $NE(S(G))$ is stable and non-empty.[21] Also, let $\overline{a}_i = \max_{\succsim_i} A_i$ and $\underline{a}_i = \min_{\succsim_i} A_i$. The following result provides necessary and sufficient conditions for an outcome to be supported with incomplete and renegotiable contracts.

Theorem 3. *Let $G \in \mathscr{G}$. An outcome (a_1^*, a_2^*) of G can be supported with incomplete and renegotiable contracts if (only if, resp.) $a_1^* \succsim_1 \overline{a}_1^{NE}$ ($a_1^* \succsim_1 \underline{a}_1^{NE}$, resp.), and $a_2^* = br_2(a_1^*)$,*

$$u_1(a_1^*, br_2(a_1^*)) \geq \max\left\{u_1(br_1(\underline{a}_2), \underline{a}_2), u_1(\overline{a}_1, br_2(\overline{a}_1))\right\} \tag{9}$$

for some selection $(br_1, br_2) \in BR_1 \times BR_2$.

In other words, in this environment outcomes in which player 1 plays an action that is "smaller" than his smallest Nash equilibrium action (in the simultaneous move version of the original game) cannot be supported. Conversely, any outcome in which player 1's action is greater than his largest Nash equilibrium action can be supported, as long as player 2 best responds to that action in a way that condition (9) is satisfied.

Also note that, if $S(G)$ has a unique Nash equilibrium, then the above proposition provides a full characterization. In many games condition (9) is trivially satisfied for any (br_1, br_2), which implies that, in this case, an outcome can be supported if and only if player 1's action is greater than his Nash equilibrium action in $S(G)$ and player 2's action is a best response to that. In fact, in such an environment, renegotiation has no bite at all. More precisely, the following can be proved by adapting the proof of Theorem 3.

Corollary 1. *Let $G \in \mathscr{G}$, and suppose that $S(G)$ has a unique pure strategy Nash equilibrium, and that $a_1 \succsim_1 a_1^{NE}$ implies $u_1(a_1, br_2(a_1)) \geq u_1(\overline{a}_1, br_2(\overline{a}_1))$ for all $br_2 \in BR_2$. If an outcome (a_1^*, a_2^*) of G can be supported with incomplete contracts, then it can also be supported with incomplete and renegotiable contracts.*

[21] Clearly, if u_2 has strictly *decreasing* differences and u_1 has *negative* externality, the game is still a member of $\mathscr{G}$.

For example, consider a *Stackelberg game* in which firm 1 moves first by choosing an output level $q_1 \in Q_1$ and firm 2, after observing q_1, chooses its own output level $q_2 \in Q_2$. We assume that Q_i, $i = 1,2$, is a finite subset of $\mathbb{R}_+$ and includes 0. Let $p : \mathbb{R}^2_+ \to \mathbb{R}_+$ be the market inverse demand function and $c_i : \mathbb{R}_+ \to \mathbb{R}_+$ be the firm i's cost function. We assume that c_i is increasing, with $c_i(0) = 0$, p is decreasing, and $p(q_1, q_2) = 0$, if $q_1 = \max Q_1$ or $q_2 = \max Q_2$. Profit function of firm i is given by $\pi_i(q_1, q_2) = p(q_1, q_2)q_i - c_i(q_i)$ and both firms are profit maximizers.

Define the game G as follows: Let $A_1 = Q_1$ and $A_2 = \{-q_2 : q_2 \in Q_2\}$ and define $\succsim_i$ on A_i as $a_i \succsim_i a'_i \Leftrightarrow a_i \geq a'_i$. Let the payoff functions be equal to the profit functions, that is

$$u_1(a_1, a_2) = p(a_1, -a_2)a_1 - c_1(a_1)$$
$$u_2(a_1, a_2) = p(a_1, -a_2)(-a_2) - c_2(-a_2)$$

for any $(a_1, a_2) \in A_1 \times A_2$. The game G is strategically equivalent to the Stackelberg game defined in the previous paragraph, and u_1 has positive externality. If we further assume that the payoff functions are single-peaked, u_2 has strictly increasing differences, and the stability condition is satisfied, then $G \in \mathscr{G}$, and hence we can apply Theorem 3.[22] Also note that, under our assumptions,

$$\max\left\{u_1(br_1(\underline{a}_2), \underline{a}_2), u_1(\overline{a}_1, br_2(\overline{a}_1))\right\} = 0.$$

Therefore, if there is a unique Nash equilibrium of the corresponding Cournot game, then an outcome can be supported if and only if firm 1 obtains non-negative profit, its output is at least as high as its Cournot Nash equilibrium output, and the follower's output is a best response to that. In such a game, therefore, firm 2 may benefit from third-party contracts, even when they are renegotiable.

As another example, consider an *ultimatum bargaining game* in which the set of possible offers is $A_1 = \{1, 2, \ldots, n\}$, for some integer $n > 1$, and $A_2 = \{Y, N\}$. Let $a_1 \succsim_1 a'_1$ if and only if $a_1 \geq a'_1$ and $Y \succ_2 N$. Suppose that if the responder (player 2) accepts an offer a_1, i.e., chooses Y, then the proposer's (player 1) payoff is $n - a_1$ and that of the responder is a_1, while if the responder rejects, i.e., chooses N, they both get zero payoff. This game satisfies all the assumptions required for Theorem 3, its simultaneous move version has a unique Nash equilibrium given by $(1, Y)$, and condition (9) is trivially satisfied. Therefore, every offer can be supported with incomplete and renegotiable contracts.

Finally, we should note that if u_1 has strictly increasing differences as well as u_2, then $S(G)$ is a supermodular game and hence it has a smallest and largest pure strategy Nash equilibria (Topkis (1979)). Furthermore, it can be shown that $NE(S(G))$ is stable in the sense of Definition 11.[23] Therefore, if G is such that u_1 and u_2 have strictly increasing differences, are single-peaked, and u_1 has positive externality, then $G \in \mathscr{G}$.

7 Strong Renegotiation-Proofness

One may object to our definition of renegotiation-proof perfect Bayesian equilibrium on the basis that off-the-equilibrium beliefs during the renegotiation process are left free. In particular, after the initial contract f and faced with an (off-the-equilibrium) renegotiation offer g, our definition allows the beliefs of the third party to assign positive probability to any action a_1. This enables us to con-

[22] For example, if the demand and cost functions are linear, then all of these conditions are satisfied.

[23] This assertion is proved in Section 9 as Lemma 10.

struct a PBE in the proof of Proposition 2 in which the initial contract f is not renegotiated as long as (f, b_2^*) is renegotiation-proof as defined in Definition 6. A plausible way to strengthen our definition of renegotiation-proof equilibrium is to require that it satisfies the intuitive criterion as defined by Cho and Kreps (1987). When applied to our setting this criterion requires that beliefs put positive probability only on actions for which it is not suboptimal to offer (g, b_2), i.e., only on those actions a_1' for which $u_2(a_1', b_2(a_1')) - g(b_2(a_1')) \geq u_2(a_1', b_2^*(a_1')) - f(b_2^*(a_1'))$. This leads to the following definition.

Definition 12 (Strong Renegotiation Proofness)**.** We say that $(f, b_2^*) \in \mathscr{C} \times A_2^{A_1}$ is *strongly renegotiation-proof* if for all $a_1 \in A_1$ for which there exists an incentive compatible $(g, b_2) \in \mathscr{C} \times A_2^{A_1}$ such that

$$u_2(a_1, b_2(a_1)) - g(b_2(a_1)) > u_2(a_1, b_2^*(a_1)) - f(b_2^*(a_1)) \tag{10}$$

and

$$g(b_2(a_1)) > f(b_2^*(a_1)) \tag{11}$$

there exists an $a_1' \in A_1$ such that

$$f(b_2^*(a_1')) \geq g(b_2(a_1')) \tag{12}$$

and

$$u_2(a_1', b_2(a_1')) - g(b_2(a_1')) \geq u_2(a_1', b_2^*(a_1')) - f(b_2^*(a_1')) \tag{13}$$

When we work with this definition, Theorem 2 needs to be modified as follows.

Theorem 4. *(f, b_2^*) is strongly renegotiation-proof if and only if for any $i \in \{1, 2, \ldots, n\}$ and $b_2 \in \mathfrak{B}(i, b_2^*)$ there exists a $k \in \{1, 2, \ldots, i-1\}$ such that*

$$u_2(a_1^i, b_2(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i)) + \sum_{j=k}^{i-1} U(b_2)_{2j-1} - \min\{0, u_2(a_1^k, b_2(a_1^k)) - u_2(a_1^k, b_2^*(a_1^k))\} \leq f_k - f_i \tag{14}$$

or there exists an $l \in \{i+1, i+2, \ldots, n\}$ such that

$$u_2(a_1^i, b_2(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i)) + \sum_{j=i+1}^{l} U(b_2)_{2(j-1)} - \min\{0, u_2(a_1^l, b_2(a_1^l)) - u_2(a_1^l, b_2^*(a_1^l))\} \leq f_l - f_i \tag{15}$$

Note that (14) and (15) are identical to their counterparts in Theorem 2 if $u_2(a_1^k, b_2(a_1^k)) \geq u_2(a_1^k, b_2^*(a_1^k))$ and $u_2(a_1^l, b_2(a_1^l)) \geq u_2(a_1^l, b_2^*(a_1^l))$. In this case, $f(b_2^*(a_1^k)) \geq g(b_2(a_1^k))$ and $f(b_2^*(a_1^l)) \geq g(b_2(a_1^l))$ imply that

$$u_2(a_1^k, b_2(a_1^k)) - g(b_2(a_1^k)) \geq u_2(a_1^k, b_2^*(a_1^k)) - f(b_2^*(a_1^k))$$

and

$$u_2(a_1^l, b_2(a_1^l)) - g(b_2(a_1^l)) \geq u_2(a_1^l, b_2^*(a_1^l)) - f(b_2^*(a_1^l))$$

Therefore, in this case a renegotiation-proof (f, b_2^*) is also strongly renegotiation-proof. If, however, there exists no $j \neq i$ such that $u_2(a_1^j, b_2(a_1^j)) \geq u_2(a_1^j, b_2^*(a_1^j))$, then a renegotiation-proof (f, b_2^*) might not be strongly renegotiation-proof.

Also, it is easy to show that Proposition 2 and Theorem 3 go through when "renegotiation-proof" is replaced with "strongly renegotiation-proof," whereas Propositions 3 and 4 go through with a minor modification similar to that made for Theorem 2.

8 Concluding Remarks

In this paper we characterized outcomes that can be supported in games with incomplete and non-renegotiable as well as renegotiable third-party contracts. We have seen that incompleteness of the contracts restricts the outcomes that can be supported, in a natural way, to those in which the second mover's strategy is increasing (Theorem 1). Renegotiation imposes further constraints on these outcomes (Theorem 2) that limit them to subgame perfect equilibrium outcomes in some games. Yet, there is a large class of games in which non-subgame perfect equilibrium outcomes can be supported even with renegotiable contracts, and hence third-party contracts still have a bite (Theorem 3). In particular, in an environment common to many economic models, such as the Stackelberg and ultimatum bargaining games, any outcome in which player 1 plays an action that is larger than his Nash equilibrium action in the simultaneous move version of the game and player 2 plays a best response can be supported with incomplete and renegotiable contracts.

There are several directions along which the current work can be extended in interesting ways. The most obvious of them is to consider more general information structures and contract spaces. One interesting possibility is to assume that the third party can observe only an outcome in some arbitrary outcome space Q and that only Q is contractible. The model is closed by assuming that there is a function $p : A_1 \times A_2 \to Q$ such that $p(q|a_1, a_2)$ is the probability of outcome q when (a_1, a_2) is played in the game. This introduces moral hazard issues into the model and might change our results in non-trivial ways. Another extension along similar lines would be a model in which player 2 has some payoff relevant information that is not available to the third-party. This is closer to a standard adverse selection model but is embedded in a strategic environment.[24] Characterization of renegotiation-proof outcomes in either of these models is left for future work.

Throughout the analysis we assumed that the original game is a finite two-stage game in which the second mover's set of actions is the same after any choice by the first mover. This allowed us to formulate incentive compatibility and renegotiation-proofness as sets of linear inequalities, which were relatively easy to manipulate and apply theorems of the alternative. A more technical extension of our work would be to consider arbitrary two-player finite extensive form games. However, adapting the methods we used in the proofs to arbitrary games is not straightforward and this extension is also left for future work.

One important aspect of our paper is its use of theorems of the alternative to characterize incentive compatibility and renegotiation-proofness. We believe that these methods have the potential to be useful for models other than games with third-party contracts, such as characterizing renegotiation-proof contracts in dynamic principal-agent models or in single-person dynamic decision making problems with time-inconsistent preferences.

9 Proofs

In the game with incomplete and non-renegotiable contracts $\Gamma(G)$, player 2 has an information set at the beginning of the game, which we identify with the null history $\emptyset$, and an information set for each $(f, a_1) \in \mathscr{C} \times A_1$. Player 1 has only one information set, given by $\mathscr{C}$, and player 3 has an information

[24] As we mentioned before, Dewatripont (1988) analyzes an example of such a model and shows that contracts can have a commitment value even under renegotiation.

set for each $f \in \mathscr{C}$. In $\Gamma_R(G)$, player 2 has additional information sets corresponding to each history (f, a_1, g, y) and (f, a_1, g, n) and player 3 has an additional information set of each $(f, g) \in \mathscr{C}^2$, which we denote by $I_3(f, g)$.

Proof of Proposition 1. **[If]** Let (b_1^*, b_2^*) be a Nash equilibrium of G and f' satisfy the conditions of the proposition. For any $b_2 \in A_2^{A_1}$, let $b_2(A_1)$ be the image of A_1 under b_2 and define

$$f^*(a_2) = \begin{cases} f'(a_2), & \text{if } a_2 \in b_2^*(A_1) \\ \max_{a_1}\{u_2(a_1, a_2) - u_2(a_1, b_2^*(a_1)) + f'(b_2^*(a_1))\}, & \text{otherwise} \end{cases}$$

for any $a_2 \in A_2$, and

$$b_{2,f}^*(a_1) = \begin{cases} b_2^*(a_1), & f = f^* \\ \in \operatorname{argmax}_{a_2} u_2(a_1, a_2) - f(a_2), & f \neq f^* \end{cases}$$

for any $f \in \mathscr{C}$ and $a_1 \in A_1$. Consider the assessment (β^*, μ^*) of $\Gamma(G)$, where $\beta_2^*[\emptyset] = f^*, \beta_1^*[\mathscr{C}] = b_1^*, \beta_2^*[f, a_1] = b_{2,f}^*(a_1)$ for all $f \in \mathscr{C}$ and $a_1 \in A_1$, and $\mu^*[\mathscr{C}](f^*) = 1$. It is easy to check that this assessment induces (b_1^*, b_2^*) and is a perfect Bayesian equilibrium of $\Gamma(G)$.

[Only if] Now, suppose that (b_1^*, b_2^*) can be supported. Then, there exists a perfect Bayesian equilibrium (β^*, μ^*) that induces (b_1^*, b_2^*), i.e., $\beta_2^*[\emptyset] = f^*, \beta_1^*[\mathscr{C}] = b_1^*, \beta_2^*[f^*, a_1] = b_2^*(a_1)$ for all $a_1 \in A_1$. The fact that (b_1^*, b_2^*) is a Nash equilibrium of G is a direct consequence of sequential rationality of players 1 and 2. We now show that f^* satisfies conditions 2 and 3 stated in Proposition 1. Suppose, in contradiction to condition 2, that $f^*(b_2^*(b_1^*)) = \alpha > 0$ and consider $f'(a_2) = \alpha/2$ for all a_2. This contract is accepted by the third party and $\beta_2[f', b_1^*] \in \operatorname{argmax}_{a_2} u_2(b_1^*, a_2)$. Therefore, offering f' yields player 2 a strictly higher expected payoff than f^*, a contradiction. Finally, sequential rationality of player 2 immediately implies condition 3. $\square$

Before we turn to the proof of Theorem 1 we introduce some notation and prove a supplementary lemma. Let the number of elements in A_1 be equal to n and order its elements so that $a_1^n \succsim_1 a_1^{n-1} \succsim_1 \cdots a_1^2 \succsim_1 a_1^1$. Let e_i be the i^{th} standard basis row vector for $\mathbb{R}^n$ and define the row vector $d_i = e_i - e_{i+1}, i = 1, 2, \ldots, n-1$. Let D be the $2(n-1) \times n$ matrix whose row $2i-1$ is d_i and row $2i$ is $-d_i$, $i = 1, \ldots, n-1$. For any $b_2 \in A_2^{A_1}$ define $U(b_2)$ as a column vector with $2(n-1)$ components, where component $2i-1$ is given by $u_2(a_1^i, b_2(a_1^i)) - u_2(a_1^i, b_2(a_1^{i+1}))$ and component $2i$ is given by $u_2(a_1^{i+1}, b_2(a_1^{i+1})) - u_2(a_1^{i+1}, b_2(a_1^i))$, $i = 1, 2, \ldots, n-1$.

Notation 1. Given two vectors $x, y \in R^n$

1. $x \geq y$ if and only if $x_i \geq y_i$, for all $i = 1, 2, \ldots, n$;

2. $x > y$ if and only if $x_i \geq y_i$, for all $i = 1, 2, \ldots, n$ and $x \neq y$;

3. $x \gg y$ if and only if $x_i > y_i$, for all $i = 1, 2, \ldots, n$.

Similarly for $\leq$, $<$, and $\ll$.

For any $b_2 \in A_2^{A_1}$ and $f \in \mathscr{C}$ let $f(b_2)$ be the column vector with n components, where i^{th} component is given by $f(b_2(a_1^i)), i = 1, 2, \ldots, n$.

It is well-known that if b_2 is increasing, then, under increasing differences, incentive compatibility reduces to local incentive compatibility.[25] We state it as a lemma for future reference.

Lemma 1. *If u_2 has increasing differences and $b_2 \in A_2^{A_1}$ is increasing in $(\succsim_1, \succsim_2)$, then for any $f \in \mathscr{C}$*

$$u_2(a_1^i, b_2(a_1^i)) - f(b_2(a_1^i)) \geq u_2(a_1^i, b_2(a_1^j)) - f(b_2(a_1^j)), \text{ for all } i, j = 1, 2, \ldots, n$$

holds if and only if

$$u_2(a_1^i, b_2(a_1^i)) - f(b_2(a_1^i)) \geq u_2(a_1^i, b_2(a_1^{i-1})) - f(b_2(a_1^{i-1})), \text{ for all } i = 2, \ldots, n,$$

and

$$u_2(a_1^i, b_2(a_1^i)) - f(b_2(a_1^i)) \geq u_2(a_1^i, b_2(a_1^{i+1})) - f(b_2(a_1^{i+1})), \text{ for all } i = 1, 2, \ldots, n-1.$$

Proof of Theorem 1. **[Only if]** Suppose that (b_1^*, b_2^*) can be supported with incomplete and non-renegotiable contracts. Then, there exists a perfect Bayesian equilibrium (β^*, μ^*) of $\Gamma(G)$ that induces (b_1^*, b_2^*), i.e., $\beta_2^*[\emptyset] = f^*, \beta_1^*[\mathscr{C}] = b_1^*, \beta_2^*[f^*, a_1] = b_2^*(a_1)$ for all $a_1 \in A_1$. Given Proposition 1 we only need to prove that b_2^* is increasing. Fix orders $(\succsim_1, \succsim_2)$ in which u_2 has strictly increasing differences. Take any $a_1, a_1' \in A_1$ and assume, without loss of generality, that $a_1 \succsim_1 a_1'$. Suppose, for contradiction, that $b_2^*(a_1') \succ_2 b_2^*(a_1)$. Sequential rationality of player 2 implies that

$$u_2(a_1, b_2^*(a_1)) - f^*(b_2^*(a_1)) \geq u_2(a_1, b_2^*(a_1')) - f^*(b_2^*(a_1'))$$
$$u_2(a_1', b_2^*(a_1')) - f^*(b_2^*(a_1')) \geq u_2(a_1', b_2^*(a_1)) - f^*(b_2^*(a_1))$$

and hence

$$u_2(a_1, b_2^*(a_1')) - u_2(a_1, b_2^*(a_1)) \leq u_2(a_1', b_2^*(a_1')) - u_2(a_1', b_2^*(a_1)),$$

contradicting that u_2 has strictly increasing differences. Therefore, b_2^* must be increasing.
[If] Let (b_1^*, b_2^*) be a Nash equilibrium of G such that b_2^* is increasing and $b_1^* = a_1^k$, for some $k = 1, 2, \ldots, n$. Given Proposition 1, all we need to prove is the existence of a contract $f \in \mathscr{C}$ such that $f(b_2^*(a_1^k)) = \delta$ and

$$u_2(a_1^i, b_2^*(a_1^i)) - f(b_2^*(a_1^i)) \geq u_2(a_1^i, b_2^*(a_1^j)) - f(b_2^*(a_1^j)), \text{ for all } i, j = 1, 2, ..., n. \tag{16}$$

By Lemma 1, (16) holds if and only if $Df(b_2^*) \leq U(b_2^*)$. Therefore, we need to show that there exists $f(b_2^*) \in \mathbb{R}^n$ such that $Ef(b_2^*) \leq V$ where

$$E = \begin{pmatrix} D \\ e_k \\ -e_k \end{pmatrix}, \quad V = \begin{pmatrix} U(b_2^*) \\ \delta \\ -\delta \end{pmatrix}$$

By Gale's theorem for linear inequalities (Mangasarian (1994), p. 33), there exists such an $f(b_2^*) \in \mathbb{R}^n$ if and only if for any $y \in \mathbb{R}_+^{2n}$, $E'y = 0$ implies $y'V \geq 0$. It is easy to show that $E'y = 0$ if and only if $y_1 = y_2, y_3 = y_4, \cdots, y_{2n-1} = y_{2n}$. Let $U(b_2^*)_i$ denote the i^{th} row of $U(b_2^*)$ and note that since b_2^* is increasing and u_2 has strictly increasing differences, $U(b_2^*)_{2i-1} + U(b_2^*)_{2i} \geq 0$, for any $i = 1, 2, \ldots, n-1$.

[25] See, for example, Bolton and Dewatripont (2005), p. 78.

Therefore,

$$y'V = \sum_{i=1}^{n-1} (U(b_2^*)_{2i-1} + U(b_2^*)_{2i}) y_{2i-1} \geq 0$$

and the proof is completed. □

Proof of Proposition 2. **[If]** Let (b_1^*, b_2^*) be a Nash equilibrium of G such that b_2^* is increasing and renegotiation-proof. This implies that there exists $f' \in \mathscr{C}$ such that (f', b_2^*) is incentive compatible and renegotiation-proof. Let $f^*(b_2^*(a_1)) = f'(b_2^*(a_1)) - f'(b_2^*(b_1^*)) + \delta$ for all $a_1 \in A_1$ and note that $f^*(b_2^*(b_1^*)) = \delta$. Furthermore, using Theorem 2, it can be easily checked that (f^*, b_2^*) is incentive compatible and renegotiation-proof. For any $f \neq f^*$ and $a_1 \in A_1$, let $b_{2,f}(a_1) \in \text{argmax}_{a_2} u_2(a_1, a_2) - f(a_2)$ and $g_{(f,a_1)} \in \text{argmax}_g u_2(a_1, b_{2,g}(a_1)) - g(b_{2,g}(a_1))$ subject to $g(b_{2,g}(a_1')) \geq f(b_{2,f}(a_1'))$ for all a_1'.

Consider the following assessment (β^*, μ^*) of $\Gamma_R(G)$: $\beta_2^*[\varnothing] = f^*$; $\beta_1^*[\mathscr{C}] = b_1^*$; $\beta_2^*[f^*, a_1] = b_{2,f}(a_1)$ for all a_1;

$$\beta_2^*[f, a_1] = \begin{cases} g_{(f,a_1)}, & \text{if } u_2(a_1, b_{2,g_{(f,a_1)}}(a_1)) - g_{(f,a_1)}(b_{2,g_{(f,a_1)}}(a_1)) > u_2(a_1, b_{2,f}(a_1)) - f(b_{2,f}(a_1)) \\ b_{2,f}(a_1), & \text{otherwise} \end{cases}$$

for any $f \neq f^*$ and a_1; $\beta_2^*[f, a_1, g, y] = b_{2,g}(a_1)$ and $\beta_{2,f}[f, a_1, g, n] = b_2^*(a_1)$ for all (a_1, f, g);

$$\beta_3^*[I_3(f^*, g)] = \begin{cases} y, & \text{if } g(b_{2,g}(a_1)) > f^*(b_2^*(a_1)) \quad \forall a_1 \\ n, & \text{otherwise} \end{cases}$$

and

$$\beta_3^*[I_3(f, g)] = \begin{cases} y, & \text{if } g(b_{2,g}(a_1)) \geq f(b_{2,f}(a_1)) \quad \forall a_1 \\ n, & \text{otherwise} \end{cases}$$

for any g and $f \neq f^*$; $\mu^*[\mathscr{C}](f^*) = 1$; For any g, $\mu^*[I_3(f^*, g)](b_1^*) = 1$ if $g(b_{2,g}(a_1)) > f^*(b_2^*(a_1))$ for all a_1 and $\mu^*[I_3(f^*, g)](a_1') = 1$ if there exists a_1' such that $f^*(b_2^*(a_1')) \geq g(b_{2,g}(a_1'))$; For any $f \neq f^*$ and g, $\mu^*[I_3(f, g)](b_1^*) = 1$ if $g(b_{2,g}(a_1)) \geq f(b_{2,f}(a_1))$ for all a_1 and $\mu^*[I_3(f, g)](a_1') = 1$ if there exists a_1' such that $f(b_{2,f}(a_1')) > g(b_{2,g}(a_1'))$. It is easy to check that this assessment induces (b_1^*, b_2^*) and is a renegotiation-proof perfect Bayesian equilibrium.

[Only if] Suppose that $\Gamma_R(G)$ has a renegotiation-proof perfect Bayesian equilibrium (β^*, μ^*) that induces (b_1^*, b_2^*). Letting $\beta_2^*[\varnothing] = f^*$, we have $\beta_1^*[\mathscr{C}] = b_1^*$, $\beta_2[f^*, a_1] = b_2^*(a_1)$ for all a_1, and $\mu^*[\mathscr{C}](f^*) = 1$. Sequential rationality of player 1 implies that

$$b_1^* \in \underset{a_1}{\text{argmax}}\, u_1(a_1, b_2^*(a_1)) \tag{17}$$

whereas that of player 2 implies $u_2(a_1, b_2^*(a_1)) - f^*(b_2^*(a_1)) \geq u_2(a_1, b_2^*(a_1')) - f^*(b_2^*(a_1'))$ for all $a_1, a_1' \in A_1$, which, together with increasing differences, implies that b_2^* is increasing.

We also claim that

$$b_2^*(b_1^*) \in \underset{a_2}{\text{argmax}}\, u_2(b_1^*, a_2) \tag{18}$$

Suppose, for contradiction, that this is not the case and let $\hat{a}_2 \in \text{argmax}_{a_2} u_2(b_1^*, a_2)$ and define $\varepsilon = u_2(b_1^*, \hat{a}_2) - u_2(b_1^*, b_2^*(b_1^*)) > 0$. Define $f'(a_2) = f^*(b_2^*(b_1^*)) + \varepsilon/2$ and note that the third party accepts

f'. Assume first that $\beta_2^*[f', b_1^*] \in A_2$, i.e., f' is not renegotiated after b_1^* and note that sequential rationality of player 2 implies that $\beta_2^*[f', b_1^*] \in \operatorname{argmax}_{a_2} u_2(b_1^*, a_2)$. Therefore, player 2's payoff under f' is

$$u_2(b_1^*, \hat{a}_2) - f^*(b_2^*(b_1^*)) - \varepsilon/2 > u_2(b_1^*, b_2^*(b_1^*)) - f^*(b_2^*(b_1^*))$$

contradicting that (β^*, μ^*) is a PBE. Now assume that f' is renegotiated after b_1^*. This implies that there exists an incentive compatible (g, b_2) such that $\beta_2^*[f', b_1^*] = g$, $\beta_3^*[f', b_1^*, g] = y$, $\beta_2^*[f', b_1^*, g, y] = b_2$.[26] Therefore, letting $b_{2,f'}(b_1^*) \in \operatorname{argmax}_{a_2} u_2(b_1^*, a_2) - f'(a_2)$,

$$\begin{aligned} u_2(b_1^*, b_2(b_1^*)) - g(b_2(b_1^*)) &\geq u_2(b_1^*, b_{2,f'}(b_1^*)) - f'(b_{2,f'}(b_1^*)) \\ g(b_2(b_1^*)) &\geq f'(b_{2,f'}(b_1^*)) \end{aligned}$$

which implies that $b_2(b_1^*) \in \operatorname{argmax}_{a_2} u_2(b_1^*, a_2)$. Player 2's payoff under f' is

$$\begin{aligned} u_2(b_1^*, b_2(b_1^*)) - g(b_2(b_1^*)) &\geq u_2(b_1^*, b_{2,f'}(b_1^*)) - f'(b_{2,f'}(b_1^*)) \\ &= u_2(b_1^*, \hat{a}_2) - f^*(b_2^*(b_1^*)) - \varepsilon/2 \\ &> u_2(b_1^*, b_2^*(b_1^*)) - f^*(b_2^*(b_1^*)) \end{aligned}$$

contradicting that (β^*, μ^*) is a PBE.

Therefore, by (17) and (18), (b_1^*, b_2^*) is a Nash equilibrium of G and b_2^* is increasing. Finally, suppose that b_2^* is not renegotiation-proof. This implies that for any contract f such that (f, b_2^*) is incentive compatible, there exists an a_1' and an incentive compatible (g, b_2) such that $u_2(a_1', b_2(a_1')) - g(b_2(a_1')) > u_2(a_1', b_2^*(a_1')) - f(b_2^*(a_1'))$ and $g(b_2(a_1)) > f(b_2^*(a_1))$ for all a_1. This implies that, in any perfect Bayesian equilibrium, after history (f, a_1') player 2 strictly prefers to renegotiate and offer g and the third party accepts it. In other words, there exists no renegotiation-proof perfect Bayesian equilibrium which induces (b_1^*, b_2^*), completing the proof. □

Proof of Theorem 2. By definition $(f, b_2^*) \in \mathscr{C} \times A_2^{A_1}$ is not renegotiation-proof if and only if there exist $i = 1, 2, \ldots, n$ and incentive compatible $(g, b_2) \in \mathscr{C} \times A_2^{A_1}$ such that $u_2(a_1^i, b_2(a_1^i)) - g(b_2(a_1^i)) > u_2(a_1^i, b_2^*(a_1^i)) - f(b_2^*(a_1^i))$ and $g(b_2(a_1^j)) > f(b_2^*(a_1^j))$ for all $j = 1, 2, \ldots, n$. For any $(f, b_2^*) \in \mathscr{C} \times A_2^{A_1}$, let $f(b_2^*) \in \mathbb{R}^n$ be a vector whose row $j = 1, 2, \ldots, n$ is given by $f(b_2^*(a_1^j))$. Note that incentive compatibility of $(g, b_2) \in \mathscr{C} \times A_2^{A_1}$ is equivalent to $Dg(b_2) \leq U(b_2)$. Therefore, $(f, b_2^*) \in \mathscr{C} \times A_2^{A_1}$ is not renegotiation-proof if and only if there exist $i = 1, 2, \ldots, n$ and $(g(b_2), b_2) \in \mathbb{R}^n \times A_2^{A_1}$ such that $Dg(b_2) \leq U(b_2)$, $u_2(a_1^i, b_2(a_1^i)) - g(b_2(a_1^i)) > u_2(a_1^i, b_2^*(a_1^i)) - f(b_2^*(a_1^i))$, and $g(b_2) \gg f(b_2^*)$. Also note that $g(b_2) \gg f(b_2^*)$ if and only if there exists an $\varepsilon \gg 0$ such that $g(b_2) = f(b_2^*) + \varepsilon$. Therefore, we have the following

Lemma 2. *$(f, b_2^*) \in \mathscr{C} \times A_2^{A_1}$ is not renegotiation-proof if and only if there exist $i = 1, 2, \ldots, n$, $b_2 \in A_2^{A_1}$, and $\varepsilon \in \mathbb{R}^n$ such that $D(f(b_2^*) + \varepsilon) \leq U(b_2)$, $\varepsilon_i < u_2(a_1^i, b_2(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i))$, and $\varepsilon \gg 0$.*

We first state a theorem of the alternative, which we will use in the sequel.

Lemma 3 (Motzkin's Theorem)**.** *Let A and C be given matrices, with A being non-vacuous. Then either*

1. *$Ax \gg 0$ and $Cx \geq 0$ has a solution x*

[26] We do not consider the case in which $\beta_3^*[f', b_1^*, g] = n$ since this is equivalent to the case $\beta_2^*[f', b_1^*] \in A_2$.

or

2. $A'y_1 + C'y_2 = 0$, $y_1 > 0$, $y_2 \geq 0$ *has a solution* y_1, y_2

   but not both.

Proof of Lemma 3. See Mangasarian (1994), p. 28. □

For any $(f, b_2^*) \in \mathscr{C} \times A_2^{A_1}$, $b_2 \in A_2^{A_1}$, and $i = 1, 2, \ldots, n$, define $V = U(b_2) - Df(b_2^*)$, $C = \begin{pmatrix} V & -D \end{pmatrix}$, and

$$A = \begin{pmatrix} I_{n+1} \\ l_i \end{pmatrix}$$

where $l_i = (u_2(a_1^i, b_2(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i)))e_1 - e_{i+1}$. Note that C and A depend on and are uniquely defined by (f, b_2^*) and (i, b_2) but we suppress this dependency for notational convenience. The following lemma uses Motzkin's Theorem to express renegotiation-proofness as an alternative.

Lemma 4. *$(f, b_2^*) \in \mathscr{C} \times A_2^{A_1}$ is renegotiation-proof if and only if for any $i = 1, 2, \ldots, n$ and $b_2 \in A_2^{A_1}$ there exist $y \in \mathbb{R}^{n+2}$ and $z \in \mathbb{R}^{2(n-1)}$ such that $A'y + C'z = 0$, $y > 0$, $z \geq 0$.*

Proof of Lemma 4. By Lemma 2, (f, b_2^*) is not renegotiation-proof if and only if there exist $i = 1, 2, \ldots, n$, $b_2 \in A_2^{A_1}$, and $\varepsilon \in \mathbb{R}^n$ such that $D(f(b_2^*) + \varepsilon) \leq U(b_2)$, $\varepsilon_i < u_2(a_1^i, b_2(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i))$, and $\varepsilon \gg 0$. This is true if and only if for some i and b_2 there exists an $x \in \mathbb{R}^{n+1}$ such that $Ax \gg 0$ and $Cx \geq 0$. To see this let $\xi > 0$ and define

$$x = \begin{pmatrix} \xi \\ \xi\varepsilon \end{pmatrix}$$

Then $D(f(b_2^*) + \varepsilon) \leq U(b_2)$ if and only if $Cx \geq 0$. Also, $\varepsilon \gg 0$ and $\varepsilon_i < u_2(a_1^i, b_2(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i))$ if and only if $Ax \gg 0$. The lemma then follows from Motzkin's Theorem. □

For any $(f, b_2^*) \in \mathscr{C} \times A_2^{A_1}$, $b_2 \in A_2^{A_1}$, and $i = 1, 2, \ldots, n$, let $U(b_2)_j$ denote the j-th row of vector $U(b_2)$ and define $\alpha_1 = 1$, $\alpha_{i+1} = u_2(a_1^i, b_2(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i))$, and

$$\begin{aligned}
\alpha_{k+1} &= \sum_{j=k}^{i-1} U(b_2)_{2j-1} + \alpha_{i+1} - f(b_2^*(a_1^k)) + f(b_2^*(a_1^i)), && \text{for } k = 1, 2, \ldots, i-1, \\
\alpha_{l+1} &= \sum_{j=i+1}^{l} U(b_2)_{2(j-1)} + \alpha_{i+1} - f(b_2^*(a_1^l)) + f(b_2^*(a_1^i)), && \text{for } l = i+1, i+2, \ldots, n, \\
\beta_j &= U(b_2)_{2j} + U(b_2)_{2j-1}, && \text{for } j = 1, 2, \ldots, n-1.
\end{aligned}$$

Again, note that α_j and β_j depend on and are uniquely defined by (f, b_2^*) and (i, b_2) but we suppress this dependency. We have the following lemma.

Lemma 5. *For any $(f, b_2^*) \in \mathscr{C} \times A_2^{A_1}$, $b_2 \in A_2^{A_1}$, and $i = 1, 2, \ldots, n$, there exist $y \in \mathbb{R}^{n+2}$ and $z \in \mathbb{R}^{2(n-1)}$ such that $A'y + C'z = 0$, $y > 0$, and $z \geq 0$ if and only if there exist $\hat{y} \in \mathbb{R}^{n+1}$ and $\hat{z} \in \mathbb{R}^{(n-1)}$ such that $\hat{y} > 0$, $\hat{z} \geq 0$, and*

$$\sum_{j=1}^{n+1} \alpha_j \hat{y}_j + \sum_{j=1}^{n-1} \beta_j \hat{z}_j = 0 \tag{19}$$

Proof of Lemma 5. Fix $(f, b_2^*) \in \mathscr{C} \times A_2^{A_1}$, $b_2 \in A_2^{A_1}$, and $i = 1, 2, \ldots, n$. First note that for any y and z, $A'y + C'z = 0$ if and only if

$$y_1 + (u_2(a_1^i, b_2(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i)))y_{n+2} + V'z = 0 \tag{20}$$

$$D'z = [A'y]_{-1} \tag{21}$$

where $[A'y]_{-1}$ is the n-dimensional vector obtained from $A'y$ by eliminating the first row. Recursively adding row 1 to row 2, row 2 to row 3, and so on, we can reduce $\begin{pmatrix} D' & [A'y]_{-1} \end{pmatrix}$ to a row echelon form and show that (21) holds if and only if

$$z_{2j-1} = z_{2j} + \sum_{k=1}^{j} y_{k+1}, \quad j = 1, 2, \ldots, i-1 \tag{22}$$

$$z_{2j} = z_{2j-1} + \sum_{k=j+1}^{n} y_{k+1}, \quad j = i, i+1, \ldots, n-1 \tag{23}$$

$$y_{n+2} = \sum_{k=1}^{n} y_{k+1} \tag{24}$$

Substituting (21)-(24) into (20) we get

$$\begin{aligned} y_1 + \alpha_{i+1} \sum_{k=1}^{n} y_{k+1} + \sum_{j=1}^{i-1} U(b_2)_{2j-1} \sum_{k=1}^{j} y_{k+1} + \sum_{j=i}^{n-1} U(b_2)_{2j} \sum_{k=j+1}^{n} y_{k+1} + \sum_{j=1}^{i-1} (U(b_2)_{2j-1} + U(b_2)_{2j}) z_{2j} \\ + \sum_{j=i}^{n-1} (U(b_2)_{2j-1} + U(b_2)_{2j}) z_{2j-1} - \sum_{k=1}^{n} (f(b_2^*(a_1^k)) - f(b_2^*(a_1^i))) y_{k+1} = 0 \end{aligned} \tag{25}$$

Therefore, $A'y + C'z = 0$ if and only if equations (22) through (25) hold. Now suppose that there exist $y \in \mathbb{R}^{n+2}$ and $z \in \mathbb{R}^{2(n-1)}$ such that $y > 0$, $z \geq 0$, and (22) through (25) hold. Define $\hat{y}_j = y_j$, for $j = 1, \ldots, n+1$ and

$$\hat{z}_j = \begin{cases} z_{2j}, & j = 1, \ldots, i-1 \\ z_{2j-1}, & j = i, \ldots, n-1 \end{cases}$$

It is easy to verify that $\hat{y} > 0$, $\hat{z} \geq 0$, and $\sum_{j=1}^{n+1} \alpha_j \hat{y}_j + \sum_{j=1}^{n-1} \beta_j \hat{z}_j = 0$.

Conversely, suppose that there exist $\hat{y} \in \mathbb{R}^{n+1}$ and $\hat{z} \in \mathbb{R}^{(n-1)}$ such that $\hat{y} > 0$, $\hat{z} \geq 0$, and (19) holds. Define $y_j = \hat{y}_j$ for $j = 1, \ldots, n+1$ and $y_{n+2} = \sum_{i=1}^{n+1} \hat{y}_j$. For any $j = 1, \ldots, i-1$, let $z_{2j-1} = \hat{z}_j + \sum_{k=1}^{j} y_{k+1}$ and $z_{2j} = \hat{z}_j$, and for any $j = i, \ldots, n-1$, let $z_{2j-1} = \hat{z}_j$ and $z_{2j} = \hat{z}_j + \sum_{k=j+1}^{n} y_{k+1}$. It is straightforward to show that $y > 0$, $z \geq 0$, and (22) through (25) hold. This completes the proof of Lemma 5. □

Lemmas 4 and 5 imply that $(f, b_2^*) \in \mathscr{C} \times A_2^{A_1}$ is renegotiation-proof if and only if for any $i \in \{1, 2, \ldots, n\}$ and $b_2 \in A_2^{A_1}$, there exist $\hat{y} \in \mathbb{R}^{n+1}$ and $\hat{z} \in \mathbb{R}^{(n-1)}$ such that $\hat{y} > 0$, $\hat{z} \geq 0$, and equation (19) holds. We can now complete the proof of Theorem 2.

[Only if] Suppose, for contradiction, that there exist $i = 1, 2, \ldots, n$ and an increasing $b_2 \in A_2^{A_1}$ such that $u_2(a_1^i, b_2(a_1^i)) > u_2(a_1^i, b_2^*(a_1^i))$, but there is no $k = 1, 2, \ldots, i-1$ such that (5) holds and no $l = i+1, \ldots, n$ such that (6) holds. This implies that $\alpha_j > 0$ for all $j = 1, \ldots, n+1$. Since u_2 has increasing differences, $\beta_j \geq 0$ for all $j = 1, \ldots, n-1$. Therefore, $\hat{y} > 0$ and $\hat{z} \geq 0$ imply that $\sum_{j=1}^{n+1} \alpha_j \hat{y}_j + \sum_{j=1}^{n-1} \beta_j \hat{z}_j > 0$, which, by Lemma 5, contradicts that (f, b_2^*) is renegotiation-proof.

[If] Fix arbitrary $i = 1, 2, \ldots, n$ and increasing $b_2 \in A_2^{A_1}$ such that $u_2(a_1^i, b_2(a_1^i)) > u_2(a_1^i, b_2^*(a_1^i))$. Suppose first that there exists a $k \in \{1,\ldots,i-1\}$ such that (5) holds. This implies that $\alpha_{i+1} > 0$ and $\alpha_{k+1} \le 0$. Let $\hat{y}_{k+1} = 1$, $\hat{y}_{i+1} = \frac{-\alpha_{k+1}}{\alpha_{i+1}} \ge 0$, and all the other $\hat{y}_j = 0$ and $\hat{z}_j = 0$. This implies that equation (19) holds and, by Lemma 5, that (f, b_2^*) is renegotiation-proof. Suppose now that there exists an $l \in \{i+1,\ldots,n\}$ such that (6) holds. Then, $\alpha_{i+1} > 0$ and $\alpha_{l+1} \le 0$. Let $\hat{y}_{l+1} = 1$, $\hat{y}_{i+1} = \frac{-\alpha_{l+1}}{\alpha_{i+1}} \ge 0$ and all the other $\hat{y}_j = 0$ and $\hat{z}_j = 0$. This, again, implies that (19) holds and that (f, b_2^*) is renegotiation-proof. □

Proof of Proposition 3. Suppose that b_2^* is renegotiation-proof and fix an $i = 1,\ldots,n$ and a $b_2^i \in \mathfrak{B}(i, b_2^*)$. For any $j = 1,\ldots,n$, let $c_j = e_i - e_j$, where e_j is the j^{th} standard basis row vector for $\mathbb{R}^n$, and define

$$E_j = \begin{pmatrix} D \\ c_j \end{pmatrix}$$

Also let

$$w_k = u_2(a_1^i, b_2^i(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i)) + \sum_{j=k}^{i-1} U(b_2^i)_{2j-1}$$
$$w_l = u_2(a_1^i, b_2^i(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i)) + \sum_{j=i+1}^{l} U(b_2^i)_{2(j-1)}$$

for any $k \in \{1,\ldots,i-1\}$ and $l \in \{i+1,\ldots,n\}$ and define

$$V_j = \begin{pmatrix} U(b_2^*) \\ -w_j \end{pmatrix}$$

Incentive compatibility of (f, b_2^*) implies that $Df(b_2^*) \le 0$. Renegotiation proofness, by Theorem 2, implies that $c_k f(b_2^*) \le -w_k$ for some $k \in \{1,\ldots,i-1\}$ or $c_l f(b_2^*) \le -w_l$ for some $l \in \{i+1,\ldots,n\}$. Suppose first that there exists a $k \in \{1,\ldots,i-1\}$ such that $c_k f(b_2^*) \le -w_k$. Then we must have $E_k f(b_2^*) \le V_k$. By Gale's theorem of linear inequalities, this implies that $x \ge 0$ and $E'x = 0$ implies $x'V_k \ge 0$. Denote the first $2(n-1)$ elements of x by y and the last element by z. It is easy to show that $E'x = 0$ implies that $y_{2j-1} = y_{2j} + z$ for $j \in \{k, k+1,\ldots,i-1\}$ and $y_{2j-1} = y_{2j}$ for $j \notin \{k, k+1,\ldots,i-1\}$. Therefore,

$$\begin{aligned} x'V_k &= \sum_{j=1}^{n-1} U(b_2^*)_{2j} y_{2j} + \sum_{j=1}^{n-1} U(b_2^*)_{2j-1} y_{2j-1} - \sum_{b_2^i} z w_k \\ &= \sum_{j=1}^{n-1} (U(b_2^*)_{2j} + U(b_2^*)_{2j-1}) y_{2j} + \sum_{k}^{i-1} z(-w_k + \sum_{j=k}^{i-1} U(b_2^*)_{2j-1}) \\ &\ge 0 \end{aligned}$$

Increasing differences imply that $-w_k + \sum_{j=k}^{i-1} U(b_2^*)_{2j-1} \ge 0$ and hence k is a blocking action.

Similarly, we can show that, if there exists an $l \in \{i+1,\ldots,n\}$ such that $c_l f(b_2^*) \le -w_l$, then l is a blocking action, and this completes the proof. □

Proof of Proposition 4. We will show that there exists an $f \in \mathscr{C}$ such that (f, b_2^*) is incentive compatible and renegotiation-proof. For any $i = 1,\ldots,n$ and $b_2^i \in \mathfrak{B}(i, b_2^*)$ pick a blocking action $m(b_2^i)$ that satisfies the conditions of the proposition. Let $c_{b_2^i} = e_i - e_{m(b_2^i)}$ for each i and $b_2^i \in \mathfrak{B}(i, b_2^*)$, and let

$\sum_i |\mathfrak{B}(i, b_2^*)| \times n$ matrix C have row $c_{b_2^i}$ corresponding to each b_2^i. Let E be given by

$$E = \begin{pmatrix} D \\ C \end{pmatrix}$$

Also let

$$w_{b_2^i} = u_2(a_1^i, b_2^i(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i)) + \mathbf{1}_{\{m(b_2^i) \le i-1\}} \sum_{j=m(b_2^i)}^{i-1} U(b_2^i)_{2j-1} + \mathbf{1}_{\{i \le m(b_2^i)-1\}} \sum_{j=i+1}^{m(b_2^i)} U(b_2^i)_{2(j-1)}$$

and $\sum_i |\mathfrak{B}(i, b_2^*)| \times 1$ vector W have row $w_{b_2^i}$ corresponding to each b_2^i. Define

$$V = \begin{pmatrix} U(b_2^*) \\ -W \end{pmatrix}$$

Observe that if $Ef(b_2^*) \le V$, then $Df(b_2^*) \le U(b_2^*)$, and hence (f, b_2^*) is incentive compatible. Furthermore, $Ef(b_2^*) \le V$ implies $W \le -Cf(b_2^*)$, and, by Theorem 2, that (f, b_2^*) is renegotiation-proof. Therefore, if we can show that there exists $f(b_2^*) \in \mathbb{R}^n$ such that $Ef(b_2^*) \le V$, the proof would be completed. By Gale's theorem of linear inequalities this is equivalent to showing $x \ge 0$ and $E'x = 0$ implies $x'V \ge 0$. Decompose x into two vectors so that the first $2(n-1)$ elements constitute y and the remaining $\sum_i |\mathfrak{B}(i, b_2^*)|$ components constitute z. Notice that for any $i = 1, \ldots, n$ and $b_2^i \in \mathfrak{B}(i, b_2^*)$ there is a corresponding element of z, which we will denote $z_{b_2^i}$.

Recursively adding row 1 to row 2, row 2 to row 3, and so on, we can reduce E' to a row echelon form and show that $E'x = 0$ if and only if

$$y_{2j-1} = y_{2j} + \sum_{b_2^i} z_{b_2^i} [\mathbf{1}_{\{m(b_2^i) \le j \le i-1\}} - \mathbf{1}_{\{i \le j \le m(b_2^i)-1\}}] \tag{26}$$

for $j = 1, \ldots, n-1$.

Let $J_- = \{j \in \{1, \ldots, n-1\} : \exists b_2^i \text{ such that } i \le j \le m(b_2^i)-1\}$ and $J_+ = \{j \in \{1, \ldots, n-1\} : \exists b_2^i \text{ such that } m(b_2^i) \le j \le i-1\}$ and note that $J_- \cap J_+ = \emptyset$. To see this, suppose, for contradiction, that there exists a $j \in J_- \cap J_+$. Therefore, there exists a b_2^i such that $i \le j \le m(b_2^i) - 1$ and $b_2^{i'}$ such that $m(b_2^{i'}) \le j \le i' - 1$. This implies that $i < i'$, $m(b_2^i) > i$, $m(b_2^{i'}) < i'$, but $m(b_2^i) > m(b_2^{i'})$, contradicting the conditions of the proposition. We can therefore write (26) as

$$y_{2j} = y_{2j-1} + \sum_{b_2^i} z_{b_2^i} \mathbf{1}_{\{i \le j \le m(b_2^i)-1\}} \tag{27}$$

for $j \in J_-$ and

$$y_{2j-1} = y_{2j} + \sum_{b_2^i} z_{b_2^i} \mathbf{1}_{\{m(b_2^i) \le j \le i-1\}} \tag{28}$$

for $j \in J_+$.

Finally note that

$$x'V = \sum_{j=1}^{n-1} U(b_2^*)_{2j} y_{2j} + \sum_{j=1}^{n-1} U(b_2^*)_{2j-1} y_{2j-1} - \sum_{b_2^i} z_{b_2^i} w_{b_2^i}$$

Substituting from (27) and (28) we obtain

$$x'V = \sum_{j\in J_-} \left[U(b_2^*)_{2j} + U(b_2^*)_{2j-1}\right] y_{2j-1} + \sum_{j\in J_+} \left[U(b_2^*)_{2j} + U(b_2^*)_{2j-1}\right] y_{2j}$$
$$+ \sum_{b_2^i} z_{b_2^i} \left[-w_{b_2^i} + \mathbf{1}_{\{m(b_2^i)\le i-1\}} \sum_{j=m(b_2^i)}^{i-1} U(b_2^*)_{2j-1} + \mathbf{1}_{\{i\le m(b_2^i)-1\}} \sum_{j=i}^{m(b_2^i)-1} U(b_2^*)_{2j} \right]$$

Increasing differences, the definition of $m(b_2^i)$, and $y,z\ge 0$ imply that $x'V\ge 0$, and the proof is completed. □

Proof of Theorem 3. Before we proceed to the proof of the theorem, we first introduce some definitions and prove an intermediate lemma.

Definition 13. For any $b_2 \in A_2^{A_1}$ we say that $i\in\{1,2,\dots,n\}$ has *right (left) deviation* at b_2 if there exists an $a_2\in A_2$ such that $a_2 \succsim_2 (\precsim_2) b_2(a_1^i)$ and $u_2(a_1^i, a_2) > u_2(a_1^i, b_2(a_1^i))$. Otherwise, we say that i has *no right (left) deviation* at b_2.

Let $BR_j(a_{-j}) = \operatorname{argmax}_{a_j} u_j(a_j, a_{-j})$, for $j = 1,2$. For any $b_2\in A_2^{A_1}$ and $i\in\{1,\dots,n\}$ that has right deviation at b_2 define

$$R(i) = \{k > i : b_2(a_1^k) \in BR_2(a_1^k) \text{ and } i<j<k \text{ implies that } j \text{ has no left deviation at } b_2\}.$$

Similarly, for $i\in\{1,\dots,n\}$ that has left deviation at b_2 let

$$L(i) = \{k < i : b_2(a_1^k) \in BR_2(a_1^k) \text{ and } k<j<i \text{ implies that } j \text{ has no right deviation at } b_2\},$$

Lemma 6. *b_2^* is renegotiation-proof if for any i_1 (i_2) that has right (left) deviation at b_2^*, $R(i_1)\neq\emptyset$ $(L(i_2)\neq\emptyset)$, and $i_1<i_2$ implies $R(i_1)\cap L(i_2)\neq\emptyset$.*

Proof of Lemma 6. Fix an $i\in\{1,\dots,n\}$ and $b_2^i\in\mathfrak{B}(i,b_2^*)$. Assume first that $b_2^i(a_1^i)\succsim_2 b_2^*(a_1^i)$ and note that $R(i)\neq\emptyset$ by assumption. Let $J=\{i+1\le j\le \min R(i)-1 : b_2^*(a_1^j)\succ_2 b_2^i(a_1^j)\}$. If $J=\emptyset$, let $m(b_2^i)=\min R(i)$ and if $J\neq\emptyset$, let $m(b_2^i)=\min J$ and note that we have

$$\sum_{j=i+1}^{m(b_2^i)} \Big(u_2(a_1^j, b_2^i(a_1^{j-1})) - u_2(a_1^j, b_2^*(a_1^{j-1})) - [u_2(a_1^{j-1}, b_2^i(a_1^{j-1})) - u_2(a_1^{j-1}, b_2^*(a_1^{j-1}))]\Big)$$
$$+ u_2(a_1^{m(b_2^i)}, b_2^*(a_1^{m(b_2^i)})) - u_2(a_1^{m(b_2^i)}, b_2^i(a_1^{m(b_2^i)})) \ge 0 \quad (29)$$

which implies that $m(b_2^i)$ is a blocking action.

Assume now that $b_2^*(a_1^i)\succsim_2 b_2^i(a_1^i)$ and note that $L(i)\neq\emptyset$. Let $J=\{\max L(i)+1\le j\le i-1 : b_2^i(a_1^j)\succ_2 b_2^*(a_1^j)\}$. If $J=\emptyset$, let $m(b_2^i)=\max L(i)$ and if $J\neq\emptyset$, let $m(b_2^i)=\max J$ and note that

$$\sum_{j=m(b_2^i)}^{i-1} \Big(u_2(a_1^{j+1}, b_2^*(a_1^{j+1})) - u_2(a_1^{j+1}, b_2^i(a_1^{j+1})) - [u_2(a_1^j, b_2^*(a_1^{j+1})) - u_2(a_1^j, b_2^i(a_1^{j+1}))]\Big)$$
$$+ u_2(a_1^{m(b_2^i)}, b_2^*(a_1^{m(b_2^i)})) - u_2(a_1^{m(b_2^i)}, b_2^i(a_1^{m(b_2^i)})) \ge 0 \quad (30)$$

which, again, implies that $m(b_2^i)$ is a blocking action.

Finally, suppose that there exist $i_1 < i_2$ such that $m(b_2^{i_1}) > i_1$ and $m(b_2^{i_2}) < i_2$. This implies that i_1 has right deviation and i_2 has left deviation at b_2^*, and hence $R(i_1) \cap L(i_2) \neq \emptyset$. But this implies that $m(b_2^{i_1}) \leq m(b_2^{i_2})$ and the proof is completed by applying Proposition 4. $\square$

We can now proceed to the proof of Theorem 3.

[If] Fix an (a_1^*, a_2^*), where $a_1^* \succsim_1 \overline{a}_1^{NE}$, and a selection (br_1, br_2) such that $a_2^* = br_2(a_1^*)$ and (9) is satisfied. Define

$$b_2(a_1) = \begin{cases} \underline{a}_2, & a_1 \prec_1 a_1^* \\ br_2(a_1^*), & a_1^* \precsim_1 a_1 \prec_1 \overline{a}_1 \\ br_2(\overline{a}_1), & a_1 = \overline{a}_1 \end{cases}$$

First, note that b_2 is increasing and $b_2(a_1^*) = a_2^*$. Second, since u_2 is single peaked, b_2 satisfies the conditions of Lemma 6 and hence is renegotiation proof. Therefore, by Proposition 2, all we need to do is to show that (a_1^*, b_2) is a Nash equilibrium of the original game G. By definition $b_2(a_1^*) \in BR_2(a_1^*)$. Condition (9) implies that $u_1(a_1^*, b_2(a_1^*)) \geq u_1(a_1, b_2(a_1))$ for all $a_1 \prec_1 a_1^*$ and $u_1(a_1^*, b_2(a_1^*)) \geq u_1(\overline{a}_1, b_2(\overline{a}_1))$. Therefore, take any a_1 such that $a_1^* \prec_1 a_1 \prec_1 \overline{a}_1$. Since $NE(S(G))$ is stable, $a_1 \succ_1 a_1^* \succsim_1 br_1(br_2(a_1^*))$, which, together with single-peakedness, implies that

$$u_1(a_1^*, b_2(a_1^*)) = u_1(a_1^*, br_2(a_1^*)) \geq u_1(a_1, br_2(a_1^*)) = u_1(a_1, b_2(a_1)).$$

Therefore, $a_1^* \in \operatorname{argmax}_{a_1} u_1(a_1, b_2(a_1))$ and hence (a_1^*, b_2) is a Nash equilibrium of G.

[Only if] Suppose that $(a_1^*, a_2^*) \in A_1 \times A_2$ can be supported with incomplete and renegotiable contracts. This, by Theorem 1, implies that there exists an increasing $b_2 \in A_2^{A_1}$ such that (a_1^*, b_2) is a Nash equilibrium of G and $b_2(a_1^*) = a_2^*$. This, in turn, implies that there exists a $br_2 \in BR_2$ such that $a_2^* = br_2(a_1^*)$.

Suppose, for contradiction, that $a_1^* \prec_1 \underline{a}_1^{NE}$. Stability of $NE(S(G))$ implies that $a_1^* \precsim_1 br_1(a_2^*)$, for any br_1. Fix a br_1 and let $a_1' = br_1(a_2^*)$. Note that $a_1' \succsim_1 a_1^*$ and $u_1(a_1^*, a_2^*) < u_1(a_1', a_2^*)$, for otherwise the game $S(G)$ would have a Nash equilibrium in which player 1's action is smaller than $\underline{a}_1^{NE}$. Therefore,

$$u_1(a_1^*, b_2(a_1^*)) = u_1(a_1^*, a_2^*) < u_1(a_1', a_2^*) = u_1(a_1', b_2(a_1^*)) \leq u_1(a_1', b_2(a_1')),$$

where the last inequality follows from positive externality. This contradicts that (a_1^*, b_2) is a Nash equilibrium of G.

Choose $br_2 \in BR_2$ such that $a_2^* = br_2(a_1^*) = b_2(a_1^*)$ and take any $br_1 \in BR_1$. Suppose, for contradiction, that (9) is not satisfied for this selection of (br_1, br_2). If $u_1(br_1(\underline{a}_2), \underline{a}_2) > u_1(a_1^*, br_2(a_1^*))$, then there exists a_1' such that $u_1(a_1', \underline{a}_2) > u_1(a_1^*, br_2(a_1^*))$. This implies that

$$u_1(a_1', b_2(a_1')) \geq u_1(a_1', \underline{a}_2) > u_1(a_1^*, br_2(a_1^*)) = u_1(a_1^*, b_2(a_1^*)),$$

where the first inequality follows from positive externality and that b_2 is increasing. This contradicts that (a_1^*, b_2) is a Nash equilibrium.

To prove that $u_1(a_1^*, br_2(a_1^*)) \geq u_1(\overline{a}_1, br_2(\overline{a}_1))$, we first prove the following lemma.

Lemma 7. *If $b_2 \in A_2^{A_1}$ is renegotiation-proof, then $\overline{a}_1$ does not have right deviation.*[27]

[27] See Definition 13.

Proof of Lemma 7. Let $a_1^n = \overline{a}_1$ and suppose, for contradiction, that a_1^n has right deviation, i.e., there exists $a_2' \succ_2 b_2(a_1^n)$ such that $u_2(a_1^n, a_2') > u_2(a_1^n, b_2(a_1^n))$. Define

$$b_2'(a_1) = \begin{cases} a_2', & a_1 = a_1^n \\ b_2(a_1), & a_1 \prec_1 a_1^n \end{cases}$$

Note that b_2' is increasing and hence incentive compatible. Also,

$$\begin{aligned} u_2(a_1^n, b_2'(a_1^n)) - u_2(a_1^n, b_2(a_1^n)) - [u_2(a_1^{n-1}, b_2'(a_1^n)) - u_2(a_1^{n-1}, b_2(a_1^n))] > 0 \\ = \sum_{j=k}^{n-1} u_2(a_1^j, b_2'(a_1^j)) - u_2(a_1^j, b_2(a_1^j)) + \sum_{j=k}^{n-2} u_2(a_1^j, b_2(a_1^{j+1})) - u_2(a_1^j, b_2'(a_1^{j+1})) \end{aligned}$$

for all $k < n$, which, by Proposition 3, contradicts that b_2 is renegotiation-proof. □

Suppose, for contradiction, that $u_1(a_1^*, br_2(a_1^*)) < u_1(\overline{a}_1, br_2(\overline{a}_1))$. Then

$$u_1(\overline{a}_1, b_2(\overline{a}_1)) \geq u_1(\overline{a}_1, br_2(\overline{a}_1)) > u_1(a_1^*, br_2(a_1^*)) = u_1(a_1^*, b_2(a_1^*)),$$

where the first inequality follows from no right deviation at $\overline{a}_1$ (Lemma 7) and positive externality. Therefore, $u_1(\overline{a}_1, b_2(\overline{a}_1)) > u_1(a_1^*, b_2(a_1^*))$, which contradicts that (a_1^*, b_2) is a Nash equilibrium of G, and the proof is completed. □

Proof of Theorem 4. By definition $(f, b_2^*) \in \mathscr{C} \times A_2^{A_1}$ is not strongly renegotiation-proof if and only if there exist $i = 1, 2, \ldots, n$ and incentive compatible $(g, b_2) \in \mathscr{C} \times A_2^{A_1}$ such that $u_2(a_1^i, b_2(a_1^i)) - g(b_2(a_1^i)) > u_2(a_1^i, b_2^*(a_1^i)) - f(b_2^*(a_1^i))$, $g(b_2(a_1^i)) > f(b_2^*(a_1^i))$, and $g(b_2(a_1^j)) - f(b_2^*(a_1^j)) > \min\{0, u_2(a_1^j, b_2(a_1^j)) - u_2(a_1^j, b_2^*(a_1^j))\}$ for all $j = 1, 2, \ldots, n$. The following lemma easily follows.

Lemma 8. *$(f, b_2^*) \in \mathscr{C} \times A_2^{A_1}$ is not strongly renegotiation-proof if and only if there exist $i = 1, 2, \ldots, n$, $b_2 \in A_2^{A_1}$, and $\varepsilon \in \mathbb{R}^n$ such that $D(f(b_2^*) + \varepsilon) \leq U(b_2)$, $0 < \varepsilon_i < u_2(a_1^i, b_2(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i))$, and $\varepsilon_j > \min\{0, u_2(a_1^j, b_2(a_1^j)) - u_2(a_1^j, b_2^*(a_1^j))\}$ for all $j = 1, 2, \ldots, n$.*

Define the matrices V and C as in the proof of Theorem 2, and define the matrix A as follows: its row 1 is e_1, row $n+2$ is l_i, and row $j+1$, for $j = 1, \ldots, n$, is given by $-\min\{0, u_2(a_1^k, b_2(a_1^k)) - u_2(a_1^k, b_2^*(a_1^k))\}e_1 + e_{j+1}$. We have the following lemma, whose proof is similar to that of Lemma 4.

Lemma 9. *$(f, b_2^*) \in \mathscr{C} \times A_2^{A_1}$ is strongly renegotiation-proof if and only if for any $i = 1, 2, \ldots, n$ and $b_2 \in A_2^{A_1}$ there exist $y \in \mathbb{R}^{n+2}$ and $z \in \mathbb{R}^{2(n-1)}$ such that $A'y + C'z = 0$, $y > 0$, $z \geq 0$.*

The rest of the proof is almost identical to that of Theorem 2, and therefore is omitted. □

Lemma 10. *If u_1 and u_2 have strictly increasing differences, then $NE(S(G))$ is stable.*

Proof of Lemma 10. Assume that u_1 and u_2 have strictly increasing differences and fix a selection (br_1, br_2). It is a standard result that br_1 and br_2 are increasing. Suppose, for contradiction, that there exists an $a_1 \succsim_1 \overline{a}_1^{NE}$ such that $br_1(br_2(a_1)) \succ_1 a_1$. Consider the sequence (a_1^t, a_2^t), $t = 0, 1, \ldots$ defined by $a_1^0 = a_1$, $a_2^t = br_2(a_1^t)$, $t = 0, 1, \ldots$, and $a_1^t = br_1(a_2^{t-1})$, $t = 1, 2, \ldots$.

We claim that there exist $i = 1, 2$ and $k = 1, 2, \ldots$ such that $a_i^k = \overline{a}_i$. We will show that if $a_i^t \prec_i \overline{a}_i$ for $i = 1, 2$ and $t = 0, 1, \ldots, k-1$, then $a_i^k \succ_i a_i^{k-1}$, $k = 1, 2, \ldots$. The claim then follows from the finiteness of A_i. Note that $a_1^0 \prec_1 \overline{a}_1$ by assumption and assume that $a_2^0 \prec_2 \overline{a}_2$. We then have $a_1^1 = br_1(a_2^0) = br_1(br_2(a_1^0)) \succ_1 a_1^0$. Since br_2 is increasing, $a_2^1 = br_2(a_1^1) \succsim_2 br_2(a_1^0) = a_2^0$. If $a_2^1 = a_2^0$, then $a_1^1 = br_1(a_2^0) = br_1(a_2^1) = a_1^2$, which implies that $(a_1^1, a_2^1) \in NE(S(G))$, contradicting that $\overline{a}_1^{NE}$ is the greatest Nash equilibrium action. Therefore, we must have $a_2^1 \succ_2 a_2^0$. This shows that the claim holds for $k = 1$. Now suppose that it holds for $k = 1, 2, \ldots, l-1$ and assume that $a_i^k \prec_i \overline{a}_i$, for $i = 1, 2$ and $k = 1, \ldots, l-1$. We then have $a_1^l = br_1(a_2^{l-1}) \succsim_1 br_1(a_2^{l-2}) = a_1^{l-1}$. If $a_1^l = a_1^{l-1}$, then $a_1^{l-1} = br_1(a_2^{l-1})$ and $a_2^{l-1} = br_2(a_1^{l-1})$, which implies that $(a_1^{l-1}, a_2^{l-1}) \in NE(S(G))$, contradicting that $\overline{a}_1^{NE}$ is the greatest Nash equilibrium action. Therefore, $a_1^l \succ_1 a_1^{l-1}$. Similarly, $a_2^l = br_2(a_1^l) \succsim_2 br_2(a_1^{l-1}) = a_2^{l-1}$. If $a_2^l = a_2^{l-1}$, then $(a_1^l, a_2^{l-1}) \in NE(S(G))$, again contradicting that $\overline{a}_1^{NE}$ is the greatest Nash equilibrium action. Therefore, $a_2^l \succ_2 a_2^{l-1}$, completing the proof.

Now, assume, without loss of generality, that $a_2^k = \overline{a}_2$ for some $k = 1, 2, \ldots$. Then, $a_1^{k+1} = br_1(a_2^k) \succsim_1 br_1(a_2^{k-1}) = a_1^k$. If $a_1^{k+1} = a_1^k$, then $(a_1^k, a_2^k) \in NE(S(G))$, contradicting that $\overline{a}_1^{NE}$ is the greatest Nash equilibrium action. If $a_1^{k+1} \succ_1 a_1^k$, $a_2^{k+1} = br_2(a_1^{k+1}) \succsim_2 br_2(a_1^k) = \overline{a}_2$, and hence $a_2^{k+1} = \overline{a}_2$. This implies that $(a_1^{k+1}, \overline{a}_2) \in NE(S(G))$, again contradicting that $\overline{a}_1^{NE}$ is the greatest Nash equilibrium action. □

References

[1] Beaudry, P. and M. Poitevin (1995), "Contract Renegotiation: A Simple Framework and Implications for Organization Theory," *Canadian Journal of Economics*, 28, 302-335.

[2] Bensaid, B. and R. J. Gary-Bobo (1993), "Commitment Value of Contracts under Renegotiation Constraints," *Econometrica*, 61, 1423-29.

[3] Bhaskar, V. (2008), "Games Played in a Contracting Environment," *Games and Economic Behavior*, forthcoming.

[4] Bolton, P. and M. Dewtripont (2005), *Contract Theory*, Cambridge: The MIT Press.

[5] Bolton, P. and D. S. Scharfstein (1990), "A Theory of Predation Based on Agency Problems in Financial Contracting," *American Economic Review*, 80, 93-106.

[6] Brander, J. A. and T. R. Lewis (1986), "Oligopoly and Financial Structure: The Limited Liability Effect," *American Economic Review*, 76, 956-970.

[7] Brander, J. A. and B. J. Spencer (1985), "Export Subsidies and International Market Share Rivalry," *Journal of International Economics*, 18, 83-100.

[8] Caillaud, B., B. Jullien, and P. Picard (1995), "Competing Vertical Structures: Precommitment and Renegotiation," *Econometrica*, 63, 621-646.

[9] Cho, I. K. and D. Kreps (1987), "Signaling Games and Stable Equilibria," *Quarterly Journal of Economics*, 102, 179-221.

[10] Dewatripont, M. (1988), "Commitment through Renegotiation-Proof Contracts with Third Parties," *Review of Economic Studies*, 55, 377-389.

[11] Eaton, J. and G. M. Grossman (1986), "Optimal Trade and Industrial Policy under Oligopoly," *Quarterly Journal of Economics*, 101, 383-406.

[12] Fershtman, C. and K. Judd (1987), "Equilibrium Incentives in Oligopoly," *American Economic Review*, 77, 927-940.

[13] Fershtman, C., K. Judd, and E. Kalai (1991), "Observable Contracts: Strategic Delegation and Cooperation," *International Economic Review*, 32, 551-559.

[14] Fershtman, C. and E. Kalai (1997), "Unobserved Delegation," *International Economic Review*, 38, 763-774.

[15] Fudenberg, D. and J. Tirole (1991), "Perfect Bayesian and Sequential Equilibrium," *Journal of Economic Theory*, 53, 236-260.

[16] Jackson, M. O. and S. Wilkie (2005), "Endogenous Games and Mechanisms: Side Payments Among Players," *Review of Economic Studies*, 72, 543-566.

[17] Katz, M. (1991), "Game-Playing Agents: Unobservable Contracts as Precommitments," *Rand Journal of Economics*, 22, 307-328.

[18] Katz, M. (2006), "Observable Contracts as Commitments: Interdependent Contracts and Moral Hazard," *Journal of Economics & Management Strategy*, 15, 685-706.

[19] Koçkesen, L. (2007), "Unobservable Contracts as Precommitments," *Economic Theory*, 31, 539-552.

[20] Koçkesen, L. and E. A. Ok (2004), "Strategic Delegation by Unobservable Incentive Contracts," *Review of Economic Studies*, 71, 397-424.

[21] Koçkesen, L., E. A. Ok, and R. Sethi (2000), "The Strategic Advantage of Negatively Interdependent Preferences," *Journal of Economic Theory*, 92, 274-299.

[22] Mangasarian, O. L. (1994) *Nonlinear Programming*, New York: McGraw-Hill.

[23] Maskin, E. and J. Tirole (1992),"The Principal-Agent Relationship with an Informed Principal, II: Common Values," *Econometrica*, 60, 1-42.

[24] Polo, M. and P. Tedeschi (2000), "Delegation Games and Side-Contracting," *Research in Economics*, 54, 101-116.

[25] Prat, A. and A. Rustichini (2003), "Games Played through Agents," *Econometrica*, 71, 989-1026.

[26] Schelling, T. (1960), *The Strategy of Conflict*, Cambridge: Harvard University Press.

[27] Sklivas, S. D. (1987), "The Strategic Choice of Managerial Incentives," *Rand Journal of Economics*, 18, 452-458.

[28] Snyder, C. M. (1996), "Negotiation and Renegotiation of Optimal Financial Contracts Under the Threat of Predation," *Journal of Industrial Economics*, 44, 325-343.

[29] Spencer, B. J. and J. A. Brander (1983), “International R & D Rivalry and Industrial Strategy,” *Review of Economic Studies*, 50, 707-722.

[30] Topkis, D. (1979), “Equilibrium Points in Nonzero-sum N-person Submodular Games,” *SIAM Journal on Control and Optimization*, 17, 773-787.

[31] Vickers, J. (1985), “Delegation and the Theory of the Firm,” *Economic Journal, Supplement*, 95, 138-147.