Ambiguity and the Business Cycle

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Introduction

- What is the role of uncertainty in driving business cycles?
- Bloom *et al.* (2012) build a general equilibrium model of the business cycle with heterogeneous firms to capture the impact of "uncertainty shocks".
  - In their framework, firms are affected by idiosyncratic and aggregate sources of uncertainty and are subject to a partial irreversibility constraint.
  - They model uncertainty as the dispersion of plant-level TFP shocks at using detailed micro Census data at the 4-digit industry level.
- An increase in uncertainty causes
  - firms to develop a wait-and-see attitude, leading to a decline in investment, output, and employment.
  - It also leads to a decline in productivity growth due to reallocation effects, as productive firms pause in expanding and unproductive firms pause in contracting.

Introduction

- In this paper, we adopt the smooth ambiguity preferences of Klibanoff, Marinacci, and Mukerji (KMM) (2005, 2009) in a production economy framework with irreversible investment.
- Following Collard *et al.* (2015), agents are unsure about the distribution of the latent variable and
  - cannot distinguish a process that has moderate persistence but high volatility, and one which is less volatile but highly persistent.
- Ambiguity aversion endogenously generates "doubt and pessimism."
  - The decisions of ambiguity averse agents may be viewed in terms of the decisions of an expected utility maximizing agent with beliefs that are more uncertain and pessimistic relative to those based on inference from actual data.

KMM Preferences

The state at date t is denoted $s^t = (s_0, s_1, \cdots, s_t)$, where $s_t \in \Upsilon_t$. The agent is uncertain about the stochastic process governing the probabilities on the event tree. This uncertainty is indexed by the parameter $\theta \in \Theta$, which denotes the set of unobservable parameters.

$$V_{s^t}(f) = u(f(s^t)) + \beta\phi^{-1}\left[\int_\Theta \phi\left(\int_{\Upsilon_{t+1}} V_{(s^t,s_{t+1})}(f)d\pi_\theta(s_{t+1}|s^t)\right)d\mu(\theta|s^t)\right],$$

where $V_{s^t}(f)$ is a recursively defined direct value function, $u(\cdot)$ characterizes attitudes towards risk, β is a discount factor, $\phi(\cdot)$ is a function characterizing the agent's ambiguity attitude, and $\mu(\cdot|s^t)$ denotes the Bayesian posterior.

Uncertainty

Uncertainty in this economy is assumed to driven by the stochastic behavior of productivity growth, $g_{A,t}$.
At time t, the process for the growth rate of the technology shock is given by

$$
\begin{aligned}
g_{A_k,t+1} &= \bar{g} + x_{k,t+1} + \sigma_{A_k}\epsilon_{A_k,t+1}, && (1)\\
x_{k,t+1} &= \rho_k x_{k,t} + \sigma_{x_k}\epsilon_{x_k,t+1}, && (2)
\end{aligned}
$$

where $(\epsilon_{A_k,t+1}, \epsilon_{x_k,t+1})' \sim N(0, I)$ for $k = h, l$. At time t, the agent has available observations on the current and past values of the growth rate of technology, $g_{A,t}$. However, the agent does not know the process generating $x_{k,t}$ and forms beliefs about it, given prior beliefs at time 0 and the observations on $g_{A,t}, g_{A,t-1}, \ldots$.

The Social Planner's Problem

The transformed indirect value function for the social planner's problem for the power-power specification is given by

$$\hat{J}(\hat{k}_t, \hat{\mu}_t) = \max_{\hat{c}_t, \hat{i}_t} \left\{ \frac{\hat{c}_t^{1-\gamma} - 1}{1-\gamma} \right.$$
$$\left. + \beta \left[E_{\hat{\mu}_t} \left(E_{x_t}(\hat{J}(\hat{k}_{t+1}, \hat{\mu}_{t+1}) \exp((1-\gamma) g_{A_k, t+1})) \right)^{1-\alpha} \right]^{\frac{1}{1-\alpha}} \right\}$$

subject to

$$\begin{aligned} &\hat{c}_t + \hat{i}_t \leq \hat{k}_t^a, \\ &\exp(g_{A,t+1}) \hat{k}_{t+1} = (1-\delta)\hat{k}_t + \hat{i}_t, \\ &\hat{i}_t \geq 0, \end{aligned}$$

and the law of motion for beliefs to be discussed below.

Beliefs

- *First-order uncertainty:* Given $x_{k,t}, \rho_k$ and observations on $(\hat{c}_t, \hat{i}_t, \hat{k}_t, \hat{y}_t)$, the probability distribution over $g_{A_k,t+1} \sim N(\bar{g} + \rho_k x_{k,t}, \sigma^2_{A_k} + \sigma^2_{x_k})$ denotes the typical first-order distribution $\pi_\theta(s_{t+1}|s^t)$ in the KMM formulation.
- *Second-order uncertainty:* Let $\hat{x}_{k,t} \equiv E[x_{k,t}|g_{A,1}, \ldots, g_{A,t}]$ denote the expectation of $x_{k,t}$, conditional on the history of growth rates up to t if the beliefs were updated assuming $\rho = \rho_k$ is the true data generating process. The agent's posterior beliefs are given by $\eta_t \times N(\hat{x}_{l,t}, \Omega_l)$ and $(1 - \eta_t) \times N(\hat{x}_{h,t}, \Omega_h)$, respectively, where $\Omega_k, k = l, h$ denotes the steady state variance associated with the Kalman filter based on the process with $\rho = \rho_k$ and η_t shows the posterior belief on ρ_l.
- The agent's beliefs are summarized by the tuple $(\hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t)$.

Updating Beliefs

The updates for $\hat{x}^{(i)}_{k,t+1}$ are obtained using the Kalman filter algorithm as follows:

$$\hat{x}^{(i)}_{k,t+1}(\varepsilon_{i,t+1}) \quad = \quad \rho_k \hat{x}_{k,t} + K_k \nu^{(i)}_{k,t+1}, \quad k = l, h, \quad i = l, h,$$

where $\nu^{(i)}_{k,t+1}, (i) = l, h, k = l, h$ are the "surprises" given by $\nu^{(i)}_{k,t+1} = g_{A_i,t+1} - \bar{g} - \rho_k \hat{x}_{k,t}$. The Kalman gain parameters are given by $K_k = \rho_k \Omega_k f_k^{-1}$, $k = l, h$, where

$$\begin{aligned} f_k &= E[(g_{A_k,t+1} - E(g_{A_k,t+1}))^2 | g_{A,1}, \ldots, g_{A,t}], \\ \Omega_k &= E[(x_{k,t+1} - \hat{x}_{k,t+1})^2 | g_{A,1}, \ldots, g_{A,t}], \end{aligned}$$

s.t. $f_k = \Omega_k + \sigma^2_{A_k}$ and Ω_k are defined as the solution to $\Omega_k = \rho_k^2 \Omega_k - \rho_k^2 \Omega_k^2 f_k^{-1} + \sigma^2_{x_k}$.

Preliminary Results: SPP

Characterize the solution to the social planner's problem (SPP) with ambiguity aversion and irreversibility

$$\hat{c}_t^{-\gamma} = \lambda_t,$$

$$\lambda_t - \varphi_t = \beta \left[E_{\hat{\mu}_t} \left(E_{x_t}(\hat{J}(\hat{k}_{t+1}, \hat{\mu}_{t+1}) \exp((1-\gamma) g_{A_k,t+1})) \right)^{1-\alpha} \right]^{\frac{\alpha}{1-\alpha}} \times$$

$$E_{\hat{\mu}_t} \left[\left(E_{x_t}(\hat{J}(\hat{k}_{t+1}, \hat{\mu}_{t+1}) \exp((1-\gamma) g_{A_k,t+1})) \right)^{-\alpha} \times \right.$$

$$\left. E_{x_t}(\hat{J}_1(\hat{k}_{t+1}, \hat{\mu}_{t+1}) \exp(-\gamma g_{A_k,t+1})) \right]$$

where λ_t is the Lagrange multiplier on the resource constraint and φ_t is the multiplier on the irreversibility constraint.

Preliminary Results: SPP

The marginal value of capital is given by

$$\hat{J}_1(\hat{k}_{t+1}, \hat{\mu}_{t+1}) = \hat{c}_{t+1}^{-\gamma} \left\{ a\hat{k}_{t+1}^{a-1} + (1-\delta)\min\left(1, \right.\right.$$
$$\left.\left. E_{\hat{\mu}_{t+1}}\left[\xi_{t+1}^{0} E_{x_{t+1}}\left(\frac{\hat{J}_1((1-\delta)\hat{k}_{t+1}, \hat{\mu}_{t+2})}{\hat{c}_{t+1}^{-\gamma}} \exp(-\gamma g_{A_k,t+2})\right)\right]\right)\right\}.$$

- The marginal value of capital accounts for the fact that the irreversibility constraint may be binding next period.
- Thus, the irreversibility constraint leads to an endogenous risk premium or an option value to wait. (See Demers, Demers and Altug, 2003.)

Preliminary Results: CE

Show the equivalence between the solution to the social planner's problem and a competitive equilibrium (CE) in which

- households own shares in firms and hold their corporate debt;
- value-maximizing firms own the capital stock and make real investment decisions, which they finance through retained earning, equity or debt;
- the CE yields an MM Theorem regarding the equivalence of equity and debt finance;

Preliminary Results: SDF

The CE also yields a stochastic discount factor (SDF) adjusted for ambiguity aversion which can be used to price all assets in equilibrium as

$$M_{t,t+1} = \zeta_t \exp((1-\gamma) g_{A,t+1}) \left(\frac{\hat{c}_{t+1}}{\hat{c}_t} \right)^{-\gamma},$$

where

$$\zeta_t = \frac{\left(E_{x_t}(\hat{v}(z_{t+1}, \hat{b}^d_{t+1}, \hat{\mu}_{t+1}) \exp((1-\gamma) g_{A,t+1})) \right)^{-\alpha}}{\left[E_{\hat{\mu}_t} \left(E_{x_t}(\hat{v}(z_{t+1}, \hat{b}^d_{t+1}, \hat{\mu}_{t+1}) \exp((1-\gamma) g_{A,t+1})) \right)^{1-\alpha} \right]^{\frac{\alpha}{1-\alpha}}}.$$

Preliminary Results: RCCE

Also characterize a recursive (complete) contingent claims equilibrium (RCCE) in which

- households make consumption decisions and choose how much wealth to carry over to next period contingent on all possible realizations of the state next period, s_{t+1}, conditional on the history of the shocks up to time t, s^t;
- value-maximizing firms own the capital stock and real investment decisions, which they finance through the issuance of state-contingent securities to households.
- derive the CCE price used to price all securities in equilibrium as

$$\begin{aligned} \psi_t &= \phi^{-1'}\left[\int_\Theta \phi\left(\int_{\Upsilon_{t+1}} \tilde{V}(a_{t+1}, s_{t+1}) d\pi_\theta(s_{t+1}|s_t)\right) d\mu(\theta|s_t)\right] \times \\ & \left[\int_\Theta \phi'\left(\int_{\Upsilon_{t+1}} \tilde{V}(a_{t+1}, s_{t+1}) d\pi_\theta(s_{t+1}|s_t)\right) d\mu(\theta|s_t)\right]. \end{aligned}$$

Numerical Solution Method

We use the method of value iteration with Chebyshev interpolation, which involves approximating the function $\hat{J}(\hat{k}_t, \hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t)$ by a parametric function whose coefficients are determined according to a minimum residual method.

$$\hat{J}(\hat{k}_t, \hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t) = \max_{\hat{c}_t, \hat{i}_t} \left\{ \frac{\hat{c}_t^{\,1-\gamma} - 1}{1-\gamma} + \right.$$

$$\beta \left[\eta_t \left(\int_{-\infty}^{\infty} \left(\int\int_{-\infty}^{\infty} \hat{J}(\hat{k}^{(l)}_{t+1}, \hat{x}^{(l)}_{h,t+1}, \hat{x}^{(l)}_{l,t+1}, \eta^{(l)}_{t+1}) \exp(g_{A_l,t+1})^{1-\gamma}) dF(\varepsilon_{l,t+1}) \right)^{1-\alpha} dF(x_{l,t}) \right) + \right.$$

$$\left. (1-\eta_t) \left(\int_{-\infty}^{\infty} \left(\int\int_{-\infty}^{\infty} \hat{J}(\hat{k}^{(h)}_{t+1}, \hat{x}^{(h)}_{h,t+1}, \hat{x}^{(h)}_{l,t+1}, \eta^{(h)}_{t+1}) \exp(g_{A_h,t+1})^{1-\gamma} dF(\varepsilon_{h,t+1}) \right)^{1-\alpha} dF(x_{h,t}) \right) \right]^{\frac{1}{1-}}$$

subject to

$$\hat{c}_t + \hat{i}_t \leq \hat{k}^a_t,$$
$$\exp(g_{A,t+1})\hat{k}_{t+1} = (1-\delta)\hat{k}_t + \hat{i}_t,$$
$$\hat{i}_t \geq 0.$$

Here $\varepsilon_{k,t+1} = (\epsilon_{x_k,t+1}, \epsilon_{A_k,t+1})'$, $k = l, h$ is a 2 by 1 vector standard normal shocks and $\eta^{(l)}_{t+1}$ is the posterior probability at time $t+1$ that the model with ρ_l is the true data generating process.

What Remains Ahead

- Derive full numerical solution using the global solution method with polynomial approximations; see Ju and Miao (2012), Jahan-Parvar and Liu (2012), Collard et al. (2015), and Liu and Zhang (2014) for examples.
- Derive a solution based on a log-linear approximation; see Backus, Ferriere and Zin (2014).
- Examine the solution to the model with and without ambiguity aversion; the latter model may be termed the *filtered* model in that agents do not know the true model generating the observations and solve their problem based on the filtered probabilities of the states.
- Relax the complete markets assumption: households may be more ambiguity averse than firms, leading to *disparate valuations* of identical random income streams.