KOÇ UNIVERSITY-TÜSİAD ECONOMIC RESEARCH FORUM
WORKING PAPER SERIES

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Working Paper 1421
December 2014

KOÇ UNIVERSITY-TÜSİAD ECONOMIC RESEARCH FORUM
Rumelifeneri Yolu 34450 Sarıyer/Istanbul

Equilibria in Second Price Auctions with Private Participation Costs

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Abstract

We study equilibria in second price auctions when bidders are independently and privately informed about both their values and participation costs and their joint distributions across bidders are not necessarily identical. We show that there always exists an equilibrium in this general setting with two dimensional types of ex ante heterogeneous bidders. We provide conditions under which the equilibrium is unique. Moreover, when the bidders are ex ante symmetric, we show that there is a unique symmetric equilibrium. We also identify sufficient conditions for existence of asymmetric equilibria.

Journal of Economic Literature Classification Number: C62, C72, D44, D61, D82.

Key Words: Two-Dimensional Types, Private Participation Costs, Second Price Auctions, Existence and Uniqueness of Equilibrium.

1 Introduction

In many auction markets, bidders often incur costs in order to participate in auctions. For instance, sellers may charge an entry fee, or require registration or pre-qualification for the

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auction. It may be costly for the bidders to prepare bids, travel to the auction site, or acquire information about the rules of the auction. In general, bidders often have opportunity costs of participating in auctions. Moreover, bidders may be privately informed about their participation costs.[1] In the presence of such participation costs, not all potential bidders are willing to participate in an auction. When analyzing bidders' behavior in auctions, participation decisions along with bidding strategies and payoffs should be endogenously determined.

In this paper, we study existence and uniqueness of equilibria in second price auctions when bidders are independently and privately informed about their values and participation costs and their joint distributions across bidders are not necessarily identical.[2] In other words, we allow for ex ante heterogeneous bidders with two-dimensional types.

Since conditional on participating, each bidder cannot do better than bidding his value in a second-price auction, we naturally restrict attention to (Bayesian-Nash) equilibria in *cutoff strategies*: a bidder participates in the auction if and only if his cost is below a cutoff (as a function of his private value.) We first convert the equilibrium conditions for a profile of cutoff strategies into a system of integral equations. We then use the Schauder-Tychonoff fixed-point theorem to show that there exists a solution to the system of integral equations. This establishes existence of equilibrium in a general environment, which includes previous studies as special cases. Next, we study the uniqueness issue. When the bidders are ex ante symmetric (i.e., bidders have the same joint distribution over bi-dimensional types), we show that there is a unique symmetric equilibrium, in which each bidder uses the same cutoff strategy. Moreover, We show that when there are only two bidders, if each bidder's value and participation cost are independently distributed (not necessarily identical across bidders), the equilibrium is unique under a mild restriction on marginal distributions. In the symmetric model, we also identify conditions under which there are asymmetric equilibria. Establishing the existence and uniqueness of the equilibrium and understanding the equilibrium structure would help analyzing many interesting issues such as auction revenues and efficiency.

Green and Laffont (1984) is the first paper to study the equilibrium bidding behavior in a

[1] Several related terms have been used in the literature, including participation cost, entry fee, entry cost, or opportunity cost. For our analysis, we do not need to distinguish between (bidder) participation costs and entry fees (charged by the seller.)

[2] Our analysis in this paper applies to standard English auctions or ascending price auctions, as long as participation costs are incurred before the bidding starts, bidding itself is costless, and participation decisions are made independently. In those auctions with independent private values whoever participates will bid his value or stay in the auction until price reaches his true value. The equilibrium conditions on participation are identical to the ones in second price auctions that we analyze in this paper.

second price auction when each bidder is privately informed about his willingness to pay for the object to be auctioned and about his participation cost. Assuming that bidders' values and participation costs are independently and jointly uniformly distributed, Green and Laffont (1984) show existence and uniqueness of a symmetric equilibrium in cutoff strategies.[3]

In a recent paper, Gal et al. (2007) study second-price procurement auctions with bi-dimensional types that are independently and identically distributed. They provide conditions under which there is a unique symmetric equilibrium and also that the auctioneer may have incentives to partially reimburse bidders for the costs that incur in preparing bids. Xu et al. (2013) study how resale affects bidding and entry behavior in a model where bidders possess two-dimensional types with entry costs binomially distributed.

When players have multi-dimensional types, the study of equilibrium behavior in auctions is usually challenging, due to the lack of a natural order on types. Even in the case of second price auctions, determining equilibrium cutoff strategies can be complex. The literature on auctions with participation costs has mostly focused on single-dimensional types, where either valuations or participation costs are commonly known.[4,5] In a related paper to this one, Tan and Yilankaya (2006) characterize the equilibrium structure in second price auctions when bidders' values are private information and participation costs are identical across bidders and common knowledge. They find conditions under which the equilibrium is unique and symmetric, as well as conditions under which in addition to the symmetric equilibrium, there exist asymmetric equilibria, despite bidders being ex ante symmetric. The existence and structure of multiple equilibria can have important implications for policy design and empirical studies on auctions.

The remainder of the paper proceeds as follows. Section 2 describes our model. Section 3 establishes existence of equilibrium. The uniqueness of equilibrium is presented in Section 4. In Section 5 we discuss multiple equilibria. Concluding remarks are provided in Section 6. All the proofs are in the Appendix.

[3]However, their proof on existence and uniqueness of a symmetric equilibrium is incomplete. Indeed, as we illustrate in footnote 9, the usual sufficient condition for applying the Banach fixed-point theorem (also known as the contraction mapping theorem) is not satisfied in this setting.

[4]The literature starts with Samuelson (1985), with Stegeman (1996) and Campbell (1998) among early contributors. See Kaplan and Sela (2006), Tan and Yilankaya (2006), Celik and Yilankaya (2009), Lu (2009), Cao and Tian (2010), Cao and Tian (2013) for some of the recent contributions.

[5]There is another strand of literature where bidders learn their values after incurring participation costs (commonly known), see, for example, McAfee and McMillan (1987), Tan (1992), Levin and Smith (1994).

2 The Setup

We consider an independent value environment with one seller and n buyers. Let $N = \{1, 2, \ldots, n\}$. The seller has an indivisible object which he values at zero. The auction format is the sealed-bid second price auction (see Vickrey, 1961). In order to submit a bid, bidder i must incur a participation cost c_i. Buyer i's value for the object, v_i, and participation cost c_i are privately and independently drawn from the distribution function $K_i(v_i, c_i)$ with support $[0, 1] \times [0, 1]$. Let $k_i(v_i, c_i)$ denote the corresponding density function. In particular, when v_i and c_i are independent, we have $K_i(v_i, c_i) = F_i(v_i)G_i(c_i)$ and $k_i(v_i, c_i) = f_i(v_i)g_i(c_i)$, where $F_i(v_i)$ and $G_i(c_i)$ are the cumulative distribution functions of bidder i's valuation and participation cost, $f_i(v_i)$ and $g_i(c_i)$ are the corresponding density functions.[6]

Each bidder knows his own value and participation cost before he makes his entrance decision and does not know others' decisions when he makes his own. If bidder i decides to participate in the auction, he incurs a participation cost c_i and submits a bid. The bidder with the highest bid wins the object and pays the second highest bid. If there is only one person in the auction, he wins the object and pays 0. If there is a tie, the allocation is determined by a fair lottery.

In this second price auction mechanism with participation costs, without loss of generality, the action set for any type of bidder is :$\{No\} \cup [0, 1]$ where "$\{No\}$" denotes not participating in the auction. Bidder i incurs the participation cost if and only if his action is different from "$\{No\}$". Bidders are risk neutral and they will compare their expected payoffs from participating with their participation costs to decide whether or not to participate. If the expected payoff from participating is less than the cost, they will not participate. Otherwise, they will participate and submit bids. Further if a bidder finds participating in this second price auction optimal, he cannot do better than bidding his true valuation.

Given the (Bayesian-Nash) equilibrium strategies of all others, a bidder's expected payoff from participating in the auction is a non-decreasing function of his valuation. Putting it differently, the maximum one would like to pay to participate in an auction is a non-decreasing function of his valuation. Therefore, we can focus on Bayesian-Nash equilibria in which each bidder uses a cutoff strategy denoted by $c_i^*(v_i)$, i.e., one bids his true valuation if his participation cost is less than some cutoff and does not participate otherwise.[7] An equilibrium strategy of

[6] When v_i or c_i takes discrete values, their density functions $f_i(v)$ and $g_i(c_i)$ are reduced to the discrete probability distribution functions, which can be represented by the Dirac delta function. The density at the discrete point is infinity.

[7] Lu and Sun (2007) showed that for any auction mechanism with participation costs, the participating and nonparticipating types of any bidder are divided by a nondecreasing and equicontinuous shutdown curve.

each bidder i is then determined by the expected payoff of participating in the auction $c_i^*(v_i)$ when his value is v_i.[8] $c_i^*(v_i)$ can be interpreted as the maximal amount that bidder i would like to pay to participate in the auction when his value is v_i. Let $b_i(v_i, c_i)$ denote bidder i's strategy. Then the bidding function can be characterized by

$$b_i(v_i, c_i) = \begin{cases} v_i & \text{if } 0 \leq c_i \leq c_i^*(v_i) \\ \text{No} & \text{otherwise.} \end{cases}$$

Remark 1 At an equilibrium, $c_i^*(v_i) > 0$ is a cost cutoff (critical) point such that individual i is indifferent from participating in the auction or not. Bidder i will participate in the auction whenever $0 < c_i \leq c_i^*(v_i)$. Note that at equilibrium, we have $c_i^*(v_i) \leq v_i$.

The description of the equilibria can be slightly different under different informational structures on $K_i(v_i, c_i)$. For example, when v_i is private information and c_i is common knowledge to all bidders, $K_i(v_i, c_i) = F_i(v_i)$ (See Campbell (1998), Stegeman (1996), Tan and Yilankaya (2006) and Cao and Tian (2013)). The equilibrium is described by a valuation cutoff v_i^* for each bidder i. Bidder i submits a bid whenever $v_i \geq v_i^*$.

3 Existence

Suppose, provisionally, there exists an equilibrium in which each bidder i uses $c_i^*(v_i)$ as his participation strategy. Then for bidder i with value v_i will participate in the auction (and submit v_i) if and only if $c_i \leq c_i^*(v_i)$. The density function of submitting the bid v_i is

$$f_{c_i^*(v_i)}(v_i) = \int_0^{c_i^*(v_i)} k_i(v_i, c_i) dc_i.$$

Remark 2 When v_i and c_i are independent, bidder i with value v_i will submit the bid v_i with probability $G_i(c_i^*(v_i))$ and stay out with probability $1 - G_i(c_i^*(v_i))$.

Let $F_{c_i^*(v_i)}(v_i)$ be the corresponding cumulative probability. There is a mass at $v_i = 0$ for $F_{c_i^*(v_i)}(v_i)$. $f_{c_i^*(v_i)}(0)$ refers to the probability (density) that bidder i does not submit a bid. For each bidder i, let the maximal bid of the other bidders be m_i. Note that, if $m_i > 0$, at least one of the other bidders participates in the auction. If $m_i = 0$, no other bidder participates in the auction.

The payoff of participating in the auction for bidder i with value v_i is given by

$$\int_0^{v_i} (v_i - m_i) d \prod_{j \neq i} F_{c_j^*}(m_i),$$

[8] In equilibrium, $c_i^*(v_i)$ depends on the distributions of all bidders' valuations and participation costs.

and thus the zero expected net-payoff condition for bidder i to participate in the auction when his valuation is v_i requires that

$$c_i^*(v_i) = \int_0^{v_i} (v_i - m_i) d \prod_{j \neq i} F_{c_j^*}(m_i).$$

Following some algebraic derivations, we have

Lemma 1 For all $i \in N$,

$$c_i^*(v_i) = \int_0^{v_i} \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] dm_i. \quad (1)$$

Remark 3 When v_i and c_i are independent, $K_i(v_i, c_i) = F_i(v_i)G_i(c_i)$ and $k_i(v_i, c_i) = f_i(v_i)g_i(c_i)$, we have

$$c_i^*(v_i) = \int_0^{v_i} \prod_{j \neq i} [1 - \int_{m_i}^1 G_j(c_j(\tau)) f_j(\tau) d\tau] dm_i.$$

Taking derivative of (1) with respect to v_i, we have

$$c_i^{*\prime}(v_i) = \prod_{j \neq i} [1 - \int_{v_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau]. \quad (2)$$

Notice that $c_i^*(0) = 0$, thus the above equation is a functional differential equation with this initial condition. In the special case, when v_i and c_i are independent,

$$c_i^{*\prime}(v_i) = \prod_{j \neq i} [1 - \int_{v_i}^1 G_j(c_j^*(\tau)) f_j(\tau) d\tau].$$

From (1) and (2), we have

Lemma 2 For al $i \in N$, $c_i^*(v_i)$ has the following properties:

(i) $c_i^*(0) = 0$.

(ii) $0 \leq c_i^*(v_i) \leq v_i$.

(iii) $c_i^{*\prime}(1) = 1$.

(iv) $\frac{dc_i^*(v_i)}{dn} < 0$.

(v) $\frac{dc_i^*(v_i)}{dv_i} \geq 0$ and $\frac{d^2 c_i^*(v_i)}{dv_i^2} \geq 0$.

(i) means that, when bidder i's value for the object is 0, the value of participating in the auction for bidder i is zero and thus the cost cutoff point for the bidder to enter the auction is zero. Then, as long as the bidder has participation cost bigger than zero, he will not participate in the auction.

(ii) means that a bidder will not be willing to pay more than his value to participate in the auction.

(iii) means that, when a bidder's value is 1, the marginal willingness to pay to enter the auction is 1. The intuition is that when his value for the object is 1, he will win the object almost surely. Then the marginal willingness to pay is equal to the marginal increase in the valuation.

(iv) states that the participation cutoff point is a nondecreasing function in the number of bidders. As the number of bidders increases, the probability to win the object will decrease, holding other things constant. More bidders will increase the competition among the bidders and thus reduce the expected payoff.

(v) states that expected payoff (of participating) is increasing and convex in valuation.

Definition 1 A cutoff curve equilibrium is an n-dimensional plane comprised of $(c_1^*(v_1), c_2^*(v_2), ...c_n^*(v_n))$ that is a solution of the following equation system:

$$(P1)\begin{cases} c_1^*(v_1) = \int_0^{v_1} \prod_{j\neq 1}[1 - \int_{m_1}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau]dm_1 \\ c_2^*(v_2) = \int_0^{v_2} \prod_{j\neq 2}[1 - \int_{m_2}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau]dm_2 \\ \vdots \\ c_n^*(v_n) = \int_0^{v_n} \prod_{j\neq n}[1 - \int_{m_n}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau]dm_n. \end{cases}$$

The above equation system is an integral equation system. From (2), the derivative of $c_i^*(v_i)$ at v_i depends not only on v_i itself, but also on the future path of $c_i^*(v_j)$ with $j \neq i$ and $v_j \geq v_i$. Besides, we have multiple variables in the equation system, which increases the difficulty to study the existence of equilibrium. To overcome this multiple variable problem, we transfer the original integral equation system to the following integral equation system

$$(P2)\begin{cases} c_1^*(v) = \int_0^{v} \prod_{j\neq 1}[1 - \int_{m_1}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau]dm_1 \\ c_2^*(v) = \int_0^{v} \prod_{j\neq 2}[1 - \int_{m_2}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau]dm_2 \\ \vdots \\ c_n^*(v) = \int_0^{v} \prod_{j\neq n}[1 - \int_{m_n}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau]dm_n. \end{cases}$$

Lemma 3 Problem (P1) and problem (P2) are equivalently solvable in the sense that
(i), if $(c_1^*(v_1), c_2^*(v_2), ..., c_n^*(v_n))$ is a solution to problem (P1), then $(c_1^*(v), c_2^*(v), ..., c_n^*(v))$ is a solution to problem (P2).
(ii), if $(c_1^*(v), c_2^*(v), ..., c_n^*(v))$ is a solution to problem (P2), then $(c_1^*(v_1), c_2^*(v_2), ..., c_n^*(v_n))$ is a solution to problem (P1).

Thus we have reduced the multiple variables functional differential equation system to a single variable functional equation system. We then have the following result on the existence of equilibrium $(c_1^*(v), c_2^*(v), ...c_n^*(v))$:

Proposition 1 (The Existence Theorem) *The integral equation system (P2) has at least one solution* $(c_1^*(v), c_2^*(v), ...c_n^*(v))$*; i.e., there is always an equilibrium in which bidder* i *uses his own cutoff curve* $c_i^*(v)$.

Note that on the right side of (P2), it is a mapping of $(c_1^*(v), c_2^*(v), ...c_n^*(v))$. In the proof, we show it is a mapping from a space to itself and establish the existence of equilibrium via the proper fixed point theorem.

Remark 4 When v_i and c_i are independent, the equilibrium is an n-dimensional plane composed of $(c_1^*(v), c_2^*(v), ...c_n^*(v))$ that is a solution of the following integral equation system:

$$(P3)\begin{cases} c_1^*(v) = \int_0^v \prod_{j\neq 1}[1 - \int_{m_1}^1 G_j(c_j^*(\tau))f_j(\tau)d\tau]dm_1 \\ c_2^*(v) = \int_0^v \prod_{j\neq 2}[1 - \int_{m_2}^1 G_j(c_j^*(\tau))f_j(\tau)d\tau]dm_2 \\ \vdots \\ c_n^*(v) = \int_0^v \prod_{j\neq n}[1 - \int_{m_n}^1 G_j(c_j^*(\tau))f_j(\tau)d\tau]dm_n. \end{cases}$$

4 Uniqueness

To investigate the uniqueness of equilibrium $c^*(v)$, we first consider the case where all bidders are ex ante homogeneous in the sense that they have the same joint distribution function of valuations and participation costs and focus on the symmetric equilibrium in which all bidders use the same cutoff curve.

(P2) can be rewritten as

$$c^*(v) = \int_0^v [1 - \int_m^1 \int_0^{c^*(\tau)} k(\tau, c)dcd\tau]^{n-1} dm, \tag{3}$$

and correspondingly we have

$$c^{*\prime}(v) = [1 - \int_v^1 \int_0^{c^*(\tau)} k(\tau, c)dcd\tau]^{n-1}, c^*(0) = 0. \tag{4}$$

We then have the following result.

Proposition 2 (Uniqueness of Symmetric Equilibrium) *Suppose that all bidders have the same distribution function* $K(v, c)$*. There is a unique symmetric equilibrium where each bidder uses the same cutoff strategy.*

Remark 5 Uniqueness of the symmetric equilibrium has been established in some special cases. In Campbell (1998) and Tan and Yilankaya (2006), there is a unique symmetric cutoff point v^s for valuation. In Kaplan and Sela (2006), there is a unique symmetric cutoff point c^* for participation cost.

Remark 6 Laffont and Green (1984) investigated the existence and uniqueness of symmetric equilibrium in a symmetric model when valuations and participation costs are uniformly distributed. However, their proof for uniqueness is incomplete.[9] Our proof of Proposition 2 avoids the application of the Contraction Mapping Theorem by using way of contradiction. In the proof, we suppose that we have two different symmetric equilibria $x(v)$ and $y(v)$. We show that $x(v) > y(v)$ for $v > v^*$ is inconsistent with $x(v^*) = y(v^*)$ and $x'(v^*) < y'(v^*)$. Our uniqueness result is for general distributions.

Note that the above proposition only shows the uniqueness of symmetric equilibrium when bidders are ex ante homogeneous. It does not exclude the possibility of asymmetric equilibrium. As shown by Stegeman (1996), Campbell (1998), Tan and Yilankaya (2006), Kaplan and Sela (2006), there are asymmetric equilibria where ex ante homogeneous bidders use different cutoff strategies. As such, generally the uniqueness of equilibrium cannot be guaranteed (we will also discuss this in the next section). However, when there are two bidders and their valuations and costs are independently distributed with mild restrictions, we have uniqueness.

To show this, write corresponding functional differential equation system as:

$$(P4)\begin{cases} c_1^{*\prime}(v) = [1 - \int_v^1 G_2(c_2^*(\tau))f_2(\tau)d\tau], c_1^*(0) = 0, \\ c_2^{*\prime}(v) = [1 - \int_v^1 G_1(c_1^*(\tau))f_1(\tau)d\tau], c_2^*(0) = 0. \end{cases}$$

[9] In their proof of Lemma 3 (34-35), they first showed the following inequality

$$a^n - b^n \leq (a-b)\frac{n}{2^n} \text{ if } a \leq \frac{1}{2},\ b \leq \frac{1}{2}$$

and claim that $a^n - b^n < (a-b)\frac{1}{2}$ for $n > 1$. After they get the following expression on page 35,

$$F(\lambda^{t+1}(\theta)) - F(\lambda^t(\theta)) = \int_0^\theta \{[1 - \int_m^1 \lambda^{t+1}(\tau)d\tau]^{n-1} - [1 - \int_m^1 \lambda^t(\tau)d\tau]^{n-1}\}dm,$$

they apply the above claim to reach

$$\|F(\lambda^{t+1}(.)) - F(\lambda^t(.))\| < 1/2\|\lambda^{t+1}(.) - \lambda^t(.)\|$$

which can not pass through since now $a = 1 - \int_m^1 \lambda^{t+1}(\tau)d\tau \geq \frac{1}{2}$ and $b = 1 - \int_m^1 \lambda^t(\tau)d\tau \geq \frac{1}{2}$ and so $a^{n-1} + ba^{n-2} + \cdots + b^{n-2}a + b^{n-1}$ is not necessary less than 1, which implies that the Banach fixed point theorem cannot be applied as they claimed. Beside that, to apply the claim they made, it is required that $n - 1 > 1$, i.e., $n > 2$. Thus, strictly speaking, their proof did not show the existence and uniqueness of equilibrium for the case of two bidders.

We then have the following result.

Proposition 3 (Uniqueness of Equilibrium) *Suppose* $n = 2$*,* $G_i(c)$ *is continuously differentiable on* $[0, 1]$*,* $\delta_i = \max_c g_i(c)$*, and* $\delta_i \int_0^1 (1 - F_i(s))ds < 1$ *for* $i = 1, 2$*. Then the equilibrium is unique.*

Remark 7 When participation costs are uniformly distributed for both bidders, $\delta_i = 1$ and $\int_0^1 (1 - F_i(s))ds < 1$ for any $F_i(.)$. Thus, in this case, independently of the distribution of valuations, the equilibrium is unique.

When bidders are ex ante homogeneous, the unique equilibrium is necessarily symmetric. Laffont and Green (1984) considered the case where both valuations and participation costs are uniformly distributed.[10] We give an example with both participation costs and valuations uniformly distributed as follows.

Example 1 Suppose $G_i(c)$ and $F_i(v)$ are both uniformly distributed on $[0, 1]$. At equilibrium we have

$$c_1^{*\prime}(v) = 1 - \int_v^1 c_2^*(\tau)d\tau,$$

$$c_2^{*\prime}(v) = 1 - \int_v^1 c_1^*(\tau)d\tau.$$

Then $c_1^{*\prime\prime}(v) = c_2^*(v)$ and $c_2^{*\prime\prime}(v) = c_1^*(v)$. Thus we have $c_1^{*(4)}(v) = c_1^*(v)$ and $c_2^{*(4)}(v) = c_2^*(v)$ with $c_1^*(0) = 0$, $c_1^{*\prime}(1) = 1$, $c_2^*(0) = 0$ and $c_2^{*\prime}(1) = 1$. One can check that the only equilibrium is $c_1^*(v) = c_2^*(v) = ae^v - ae^{-v}$, where $a = \frac{e}{e^2+1}$.

5 Asymmetric Equilibria

In this section we briefly discuss the possibility of multiple equilibria when both participation costs and valuations are private information.[11]

[10] Following their proof of Lemma 3, when $n = 2$,

$$|F(\lambda^{t+1}(\theta)) - F(\lambda^t(\theta))| = |\int_0^\theta \int_m^1 [\lambda^t(\tau) - \lambda^{t+1}(\tau)]d\tau dm| \leq \int_0^\theta \int_m^1 |\lambda^t(\tau) - \lambda^{t+1}(\tau)|d\tau dm < \|\lambda^t(.) - \lambda^{t+1}(.)\| \int_0^\theta dm.$$

Thus,

$$\|F(\lambda^{t+1}(.)) - F(\lambda^t(.))\| < \|\lambda^t(.) - \lambda^{t+1}(.)\|.$$

The Contraction Mapping Theorem can be applied to show the uniqueness of equilibrium without using the claim they made ($a^n - b^n < (a - b)\frac{1}{2}$ for $n > 1$) at the beginning of their proof. The above statement can be treated as a special case for our proof to Proposition 3.

[11] See Stegeman (1996), Campbell (1998), Tan and Yilankaya (2006), Cao and Tian (2013) and Kaplan and Sela (2006) for multiplicity when costs or valuations are commonly known.

Suppose the support of v_i and c_i to be $[v_l, v_h] \times [c_l, c_h]$, where $[v_l, v_h] \times [c_l, c_h]$ is a subset of $[0, 1] \times [0, 1]$. We can extend the supports of distributions to be $[0, 1] \times [0, 1]$. Then, by Proposition 1, an equilibrium exists. However, the uniqueness of the equilibrium cannot be guaranteed. Even when bidders are ex ante homogeneous, asymmetric equilibria may exist and thus we have multiple equilibria. To see this, we assume that there are only two bidders and v_i and c_i are identically and independently distributed on the corresponding supports. The distribution functions are $F(.)$ and $G(.)$.

In one special type of asymmetric equilibrium, one of the bidders never participates (independent of his valuation) in the auction. This can happen when the support of participation costs has non-zero lower bound.

The expected payoff of participating in the auction is a non-decreasing function of one's true value. Thus the necessary and sufficient condition for a bidder to never participate is that when his value is 1, participating in the auction still gives him an expected payoff that is less than the minimum participation cost, c_l, given the strategies of other bidders. Assuming $[v_l, v_h] = [0, 1]$, we have the following result.

Proposition 4 *A necessary and sufficient condition to have an asymmetric equilibrium in which one bidder never participates is*

$$F(c_l) + \int_{c_l}^{c_h} [(1 - xG(x)]dF(x) + \int_{c_h}^{1} (1 - x)dF(x) < c_l. \tag{5}$$

Based on (5) we provide simple sufficient conditions for an asymmetric equilibrium to arise. Define $\lambda(c) = c - cF(c) - \int_c^1 F(x)dx$ with $c \in [0, 1]$. Note that when $c_l = c_h = c$, (5) becomes $\lambda(c) > 0$. This motivates us to introduce the following lemma.

Lemma 4 For any strictly convex $F(.)$ with support $[0, 1]$, there exists a unique $\bar{c} \in (0, 1)$ such that $\lambda(c) > 0$ if and only if $c \in (\bar{c}, 1)$.

Corollary 1 When $F(.)$ is strictly convex and $c_h - c_l$ is sufficiently small with $c_l > \bar{c}$, there exists an asymmetric equilibrium in which one bidder never participates and the other bidder participates whenever $v \geq c$.

It is possible that in an equilibrium, one bidder never participates and the other always participates. To see that this is possible, suppose the c_i is distributed on $[c_l, c_h]$ with distribution $G(c_i)$ and v_i is distributed on $[v_l, v_h]$ with distribution $F(v_i)$, assuming $v_h > v_l > c_h > c_l$. We give the following result.

Proposition 5 *A sufficient condition to have an asymmetric equilibrium in which bidder 1 always participates and bidder 2 never participates is*

$$\int_{v_l}^{v_h} F(v_1)dv_1 < c_l.$$

One sufficient condition for this to be true is $v_h - v_l < c_l$, which is independent of the distributions of valuations and participation costs.

6 Conclusion

This paper studies existence and uniqueness of equilibrium in second price auctions when bidders' values and participation costs are both private information. We show that under general distribution functions, there always exists an equilibrium in which each bidder uses a cutoff strategy - his maximum willingness to pay for the object as a function of his value. When bidders are ex ante homogeneous, there is a unique symmetric equilibrium. Moreover, when there are only two bidders, we provide a sufficient condition for the uniqueness of the equilibrium. Future research may be focused on identifying sufficient conditions to guarantee uniqueness in general environments.

We also show that multiple equilibria can easily arise. Specifically, in the symmetric model with two bidders, we identify sufficient conditions under which asymmetric equilibria exist. Multiplicity of equilibria has important consequence for both efficiency and seller's revenue. For instance, asymmetric equilibria are ex post inefficient: A bidder may obtain the object even when there is another bidder with a higher valuation and a lower cost, who stays out of the auction. Moreover, revenue-maximizing and ex-ante efficient auctions may be asymmetric even in a symmetric environment. This is suggested by Celik and Yilankaya (2009), who provide (separate) sufficient conditions for these to happen when participation cost is commonly known.[12] It would be interesting to study revenue-maximizing or efficient auctions when both values and participation costs are bidders' private information.

[12] Also see Stegeman (1996) for an example of an asymmetric ex-ante efficient auction.

Appendix

Proof of Lemma 1:

If $m_i = 0$, none of the other bidders will participate, the probability of which is

$$\prod_{j\neq i} F_{c_j^*}(0) = \prod_{j\neq i}\int_0^1\int_{c_j^*(\tau)}^1 k_j(\tau,c_j)dc_jd\tau.$$

Otherwise, at least one other bidder submits a bid. Then

$$\prod_{j\neq i} F_{c_j^*}(m_i) = \prod_{j\neq i}[1-\int_{m_i}^1\int_0^{c_j^*(\tau)} k_j(\tau,c_j)dc_jd\tau].$$

Thus, the cutoff curve for individual i, $i\in N$, can be characterized by

$$c_i^*(v_i) = \int_0^{v_i}(v_i-m_i)d\prod_{j\neq i}[1-\int_{m_i}^1\int_0^{c_j^*(\tau)} k_j(\tau,c_j)dc_jd\tau] + v_i\prod_{j\neq i}[\int_0^1\int_{c_j^*(\tau)}^1 k_j(\tau,c_j)dc_jd\tau].$$

Using integration by parts, we have

$$\begin{aligned} c_i^*(v_i) &= \int_0^{v_i}(v_i-m_i)d\prod_{j\neq i}[1-\int_{m_i}^1\int_0^{c_j^*(\tau)} k_j(\tau,c_j)dc_jd\tau] + v_i\prod_{j\neq i}[\int_0^1\int_{c_j^*(\tau)}^1 k_j(\tau,c_j)dc_jd\tau] \\ &= (v_i-m_i)\prod_{j\neq i}[1-\int_{m_i}^1\int_0^{c_j^*(\tau)} k_j(\tau,c_j)dc_jd\tau]\mid_0^{v_i} + v_i\prod_{j\neq i}[\int_0^1\int_{c_j^*(\tau)}^1 k_j(\tau,c_j)dc_jd\tau] \\ &\quad +\int_0^{v_i}\prod_{j\neq i}[1-\int_{m_i}^1\int_0^{c_j^*(\tau)} k_j(\tau,c_j)dc_jd\tau]dm_i \\ &= -v_i\prod_{j\neq i}[1-\int_0^1\int_0^{c_j^*(\tau)} k_j(\tau,c_j)dc_jd\tau] + v_i\prod_{j\neq i}[\int_0^1\int_{c_j^*(\tau)}^1 k_j(\tau,c_j)dc_jd\tau] \\ &\quad +\int_0^{v_i}\prod_{j\neq i}[1-\int_{m_i}^1\int_0^{c_j^*(\tau)} k_j(\tau,c_j)dc_jd\tau]dm_i. \end{aligned}$$

Since

$$\int_0^1\int_0^{c_j^*(\tau)} k_j(\tau,c_j)dc_jd\tau + \int_0^1\int_{c_j^*(\tau)}^1 k_j(\tau,c_j)dc_jd\tau = \int_0^1\int_0^1 k_j(\tau,c_j)dc_jd\tau = 1,$$

we have

$$c_i^*(v_i) = \int_0^{v_i}\prod_{j\neq i}[1-\int_{m_i}^1\int_0^{c_j^*(\tau)} k_j(\tau,c_j)dc_jd\tau]dm_i.$$

Proof of Lemma 2:

(i) Letting $v_i = 0$ in the expression of $c_i^*(v_i)$, we have the result.

(ii) Since

$$c_i^*(v_i) = \int_0^{v_i}\prod_{j\neq i}[1-\int_{m_i}^1\int_0^{c_j^*(\tau)} k_j(\tau,c_j)dc_jd\tau]dm_i \leqslant \int_0^{v_i} dv_i = v_i$$

by the nonnegativity of $\int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j$ and

$$\int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau \leq \int_{m_i}^1 \int_0^1 k_j(\tau, c_j)dc_j d\tau \leq \int_0^1 \int_0^1 k_j(\tau, c_j)dc_j d\tau = 1,$$

we have $0 \leq c_i^*(v_i) \leq v_i$.

(iii) Letting $v_i = 1$ in (2), we have the result.

(iv) As n increases, say, from n to $n+1$, the product term inside the integral will be increased by one more term. Also, note that $0 < 1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau < 1$. So as n increases, $c_i^*(v_i)$ will decrease.

(v)

$$\frac{dc_i^*(v_i)}{dv_i} = \prod_{j \neq i}[1 - \int_{v_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau] \geq 0$$

by noting that

$$\int_{v_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau \leq \int_0^1 \int_0^1 k_j(\tau, c_j)dc_j d = 1.$$

We then have

$$\frac{d^2 c_i^*(v_i)}{dv_i^2} = \sum_{k \neq i} \prod_{j \neq i, j \neq k} [1 - \int_{v_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau] \int_0^{c_k^*(v_i)} k_j(\tau, c_j)dc_j d\tau \geq 0.$$

Proof of Lemma 3:

Suppose $(c_1^*(v_1), c_2^*(v_2), ..., c_n^*(v_n))$ is a solution to problem (P1). Then we have for any $i \in N$,

$$c_i^*(v_i) = \int_0^{v_i} \prod_{j \neq i}[1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau]dm_i.$$

By changing the variable v_i to v we have

$$c_i^*(v) = \int_0^{v} \prod_{j \neq i}[1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau]dm_i$$

for all $i \in N$. So $(c_1^*(v), c_2^*(v), ..., c_n^*(v))$ is a solution to (P2).

Now suppose $(c_1^*(v), c_2^*(v), ..., c_n^*(v))$ is a solution to (P2), then we have for any $i \in N$,

$$c_i^*(v) = \int_0^{v} \prod_{j \neq i}[1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau]dm_i.$$

Then by changing the variable v to v_i in the i^{th} equation we have

$$c_i^*(v_i) = \int_0^{v_i} \prod_{j \neq i}[1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau]dm_i.$$

Thus $(c_1^*(v_1), c_2^*(v_2), ..., c_n^*(v_n))$ is a solution to (P1).

Proof of Proposition 1:

Let $h_i(m_i, c^*) = \prod_{j \neq i}[1 - \int_{m_i}^{1} \int_{0}^{c_j^*(\tau)} k_j(\tau, c) dc d\tau]$. Since $k_j(\tau, c)$ is integrable over c as it is a density function, there exists a continuous function $\gamma_j(\tau, c)$ with $\frac{\partial \gamma_j(\tau,c)}{\partial c} = k_j(\tau, c)$ such that $\int_0^{c_j^*(\tau)} k_j(\tau, c)dc = \gamma_j(\tau, c_j^*(\tau)) - \gamma_j(\tau, 0)$. Thus $h_i(m_i, c^*) = \prod_{j \neq i}[1 - \int_{m_i}^{1}[\gamma_j(\tau, c_j^*(\tau)) - \gamma_j(\tau, 0)]d\tau]$, which is a continuous mapping from $[0,1] \times [0,1]^n \rightarrow [0,1]$.

Let $H(m, c^*) = (h_1(m_1, c^*), h_2(m_2, c^*), \cdots, h_n(m_n, c^*))'$, which is a continuous mapping from $[0,1]^n \times [0,1]^n \rightarrow [0,1]^n$. By Lemma 2, H is bounded above by one. Define

$$M = \{c \in \varphi \mid : \|c\| \leq 1\},$$

where φ is the space of continuous function ϕ defined on $[0,1]^n \rightarrow [0,1]^n$ with $\|c\| = \sup_{0 \leq v \leq 1} c(v)$. Then by Ascoli Theorem, M is compact. M is clearly convex. Define an operator $P : M \rightarrow M$ by

$$(Pc)(v) = \int_0^v H(s, c(.))ds$$

Then, by Lemma 2, P is a continuous function from M to itself. Thus, by Schauder-Tychonoff Fixed-point Theorem, there exists a fixed point; i.e., a solution for the functional differential equation system exists.

Proof of Proposition 2:

The existence of the symmetric equilibrium can be established by the Schauder-Tychonoff Fixed-point Theorem. Here we only need to prove the uniqueness of the symmetric equilibrium. Suppose, by way of contradiction, that we have two different symmetric equilibria $x(v)$ and $y(v)$. Then we have

$$x'(v) = [1 - \int_v^1 \int_0^{x(\tau)} k(\tau, c)dcd\tau]^{n-1}$$
$$y'(v) = [1 - \int_v^1 \int_0^{y(\tau)} k(\tau, c)dcd\tau]^{n-1}.$$

Suppose $x(1) > y(1)$, then by the continuity of $x(v)$ and $y(v)$ we can find a v^* such that $x(v^*) = y(v^*) = c(v^*)$ and $x(v) > y(v)$ for all $v \in (v^*, 1]$ by noting that $x(0) = y(0)$.

Case 1: If $k(v, c) > 0$ with positive probability measure on $(v^*, 1) \times (c(v^*), 1)$, then for $\tau \in (v^*, 1]$ we have

$$\int_0^{x(\tau)} k(\tau, c)dc > \int_0^{y(\tau)} k(\tau, c)dc$$

for $\tau \in (v^*, 1)$. Then we have $x'(v^*) < y'(v^*)$ which is a contradiction to $x(v) > y(v)$ for $v > v^*$. So we have $x(1) = y(1)$.

Now suppose there exists an interval $[\alpha,\beta]\subset[0,1]$ such that $x(\alpha)=y(\alpha)$ and $x(\beta)=y(\beta)$ while for all $v\in(\alpha,\beta)$, $x(v)>y(v)$ and for all $v\in[\beta,1]$, $x(v)=y(v)$, by the same logic above, we have $x(\beta)=y(\beta)$ and $x'(v)<y'(v)$ for $v\in(\alpha,\beta)$, which is inconsistent with $x(v)>y(v)$ for all $v\in(\alpha,\beta)$. Thus we can prove that $x(v)=y(v)$ for all $v\in[0,1]$ and so the symmetric equilibrium is unique.

Case 2: If $k(v,c)>0$ with zero probability measure on $(v^*,1)\times(c(v^*),1)$, then we have $x'(v)=y'(v)$ for all $v\in(v^*,1]$. By $x(v^*)=y(v^*)$ we have $x(v)=y(v)$ for all $v>v^*$, which is a contradiction to $x(v)>y(v)$. Thus there is a unique symmetric equilibrium.

Then in both cases we prove that there is a unique symmetric equilibrium.

Proof of Proposition 3:

Define a mapping

$$(Pc)(v)=\int_0^v ds-\int_0^v\int_s^1\begin{pmatrix}0 & f_2(\tau)\\ f_1(\tau) & 0\end{pmatrix}\begin{pmatrix}G_1(c_1(\tau))\\ G_2(c_2(\tau))\end{pmatrix}d\tau ds,$$

where $c=(c_1,c_2)'$.

Take any $x(v)=(x_1(v),x_2(v))'$ and $y(v)=(y_1(v),y_2(v))'$ with $x(v)$, $y(v)\in\varphi$ where φ is the space of monotonic increasing continuous functions defined on $[0,1]\to[0,1]$. For presentation convenience, denote

$$h(\widehat{x}_1(\tau),\widehat{x}_2(\tau),\tau)=\begin{pmatrix}0 & g_2(\widehat{x}_2(\tau))f_2(\tau)\\ g_1(\widehat{x}_1(\tau))f_1(\tau) & 0\end{pmatrix}.$$

Then we have

$$\begin{aligned}|(Px)(v)-(Py)(v)| &\leq \int_0^v\int_s^1 h(\widehat{x}_1(\tau),\widehat{x}_2(\tau),\tau)|\begin{pmatrix}x_1(\tau)-y_1(\tau)\\ x_2(\tau)-y_2(\tau)\end{pmatrix}|d\tau ds\\ &= \int_0^v\int_s^1 h(\widehat{x}_1(\tau),\widehat{x}_2(\tau),\tau)d\tau ds\sup_{0<v\leq 1}|x(v)-y(v)|\\ &\leq \int_0^1\int_s^1 h(\widehat{x}_1(\tau),\widehat{x}_2(\tau),\tau)d\tau ds\sup_{0<v\leq 1}|x(v)-y(v)|\\ &\leq \int_0^1\begin{pmatrix}0 & \delta_2(1-F_2(s))\\ \delta_1(1-F_1(s)) & 0\end{pmatrix}ds\sup_{0<v\leq 1}|x(v)-y(v)|, \quad (6)\end{aligned}$$

where the first equality comes from mean value theorem, $\widehat{x}_i(\tau)$ is some number between $x_i(\tau)$ and $y_i(\tau)$, and δ_i is the maximum of $g_i(c)$, $i=1,2$. Thus when $\delta_i\int_0^1(1-F_i(s))ds<1$, the above mapping is a contraction, so there exists a unique equilibrium.

Proof of Proposition 4:

We first prove *necessity*. Suppose, in an asymmetric equilibrium, bidder 2 never participates, then bidder 1 participates if and only if $v_1 \geq c_1$ and thus we have $c_1^*(v_1) = v_1$. Given this, the expected revenue of bidder 2 with $v_2 = 1$ when he participates in the auction is h, where

$$h = F(c_l) + \int_{c_l}^{c_h} [(1-x)G(x) + (1 - G(x))]dF(x) + \int_{c_h}^{1} (1-x)dF(x).$$

The first term is bidder 2's expected payoff when bidder 1's value is less than c_l (with probability $F(c_l)$). In this case, bidder 2 does not participate and bidder 2 will get payoff 1. The second term is the payoff when bidder 1's value is between c_l and c_h. For any $v_1 \in (c_l, c_h)$, bidder 2's payoff is $1 - v_1$ when bidder 1 participates and is 1 when bidder 1 does not participate, and the probabilities are $G(v_1)$ and $1 - G(v_1)$ separately. The third term is the payoff when $v_1 \geq c_h$ and in this case bidder 1 participates for sure. For bidder 2 never participating, we need $h < c_l$, which can be simplified to (5) and thus we have necessity.

Next we prove *sufficiency*. Suppose (5) holds. Consider the strategies that bidder 2 never participates and bidder 1 participates whenever $v_1 \geq c_1$. Given the strategy of bidder 2, bidder 1's best response is to participate whenever $v_1 \geq c_1$. Given the strategy of bidder 1, since (5) holds, the expected revenue of bidder 2 with $v_2 = 1$ is less than c_l, thus the best response for bidder 2 is never to participate for any type. Thus there exists an asymmetric equilibrium in which one bidder never participates and we have sufficiency.

Proof of Lemma 4:

To see this, note that from $\lambda(c)$, we have $\lambda(0) = -\int_0^1 F(x)dx < 0$, $\lambda(1) = 0$ and $\lambda'(c) = 1 - cf(c)$. For strictly convex $F(.)$, $\lambda(c)$ is a concave function of c with $\lambda(1) = 0$ since $\lambda''(c) = -f(c) - cf'(c) < 0$. Also note that $\lambda'(1) = 1 - f(1) < 0$. By the strict concavity of $\lambda(c)$ with $\lambda(1) = 0$, there exists a unique $\bar{c}$ such that $\lambda(c) > 0$ if and only if $c \in (\bar{c}, 1)$.

Proof of Corollary 1:

By Lemma 4, we have $\lambda(c) > 0$ for $c \in (\bar{c}, 1)$. By continuity, when $c_h - c_l$ is sufficiently small,

$$c_l - h > 0$$

for all $c_l \in (\bar{c}, 1)$, where where h is the expected revenue of the bidder 2 with value equal to 1 when he participates in the auction. Thus (5) holds and an asymmetric equilibrium in which one bidder never participates exists. The other bidder participates whenever $v \geq c$.

Proof of Proposition 5:

Suppose we have an equilibrium in which bidder 1 always participates and bidder 2 never participates. Then bidder 1 always participates is a best response to bidder 2's strategy. For bidder 2's strategy to be a best response, we need

$$\int_{v_l}^{v_h} (v_h - v_1) dF(v_1) - c_l < 0,$$

the expected payoff for bidder 2 with value $v_2 = v_h$ is less than the lowest participation cost. Using integration by parts we have

$$\int_{v_l}^{v_h} F(v_1) dv_1 < c_l.$$

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